Max-flow via LP:

Variables f_e for every $e \in E$

objective function: max $\sum_{\overrightarrow{sz} \in E} f_{sz}$

constraints:

for every $e \in E$, $0 \leq f_e \leq c_e$

for every $v \in V - \{s,t\}$, $\sum_{\overrightarrow{wv} \in E} f_{wv} = \sum_{\overrightarrow{vz} \in E} f_{vz}$

Standard form for LPs:

Variables $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ m constraints $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

objective $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

LP example from last class:

$$\max \quad x_1 + 6x_2 + 10x_3$$

s.t.

$$x_1 \leq 300 \quad (1)$$
$$x_2 \leq 200 \quad (2)$$
$$x_1 + 3x_2 + 2x_3 \leq 1000 \quad (3)$$
$$x_2 + 3x_3 \leq 500 \quad (4)$$
$$x_1, x_2, x_3 \geq 0 \quad (5a, 5b, 5c)$$

Optimal is at $(200, 200, 100)$ which has profit $= 2400$

How do we know it's optimal?

Let $y = (y_1, y_2, y_3, y_4) = (0, \frac{1}{3}, 1, \frac{8}{3})$

Then $y_1 \times (1) + y_2 \times (2) + y_3 \times (3) + y_4 \times (4)$

$$\Leftrightarrow \quad x_1 + 6x_2 + 10x_3 \leq 2400$$

So profit ≤ 2400. ✓

How do we find this y?

$$y_1 \times (1) + y_2 \times (2) + y_3 \times (3) + y_4 \times (4)$$

$$\Leftrightarrow x_1 y_1 + x_2 y_2 + x_1 y_3 + 3x_2 y_3 + 2x_3 y_3 + x_2 y_4 + 3x_3 y_4$$
$$\le 300y_1 + 200y_2 + 1000y_3 + 500y_4$$

$$\Leftrightarrow x_1(y_1 + y_3) + x_2(y_2 + 3y_3 + y_4) + x_3(2y_3 + 3y_4)$$
$$\le 300y_1 + 200y_2 + 1000y_3 + 500y_4$$

goal: this is $\ge x_1 + 6x_2 + 10x_3$

goal: minimize this quantity

which means

$$y_1 + y_3 \ge 1$$
$$y_2 + 3y_3 + y_4 \ge 6$$
$$2y_3 + 3y_4 \ge 10$$

Dual LP:

$$\min \; 300y_1 + 200y_2 + 1000y_3 + 500y_4$$

s.t.

$$y_1 + y_3 \ge 1$$
$$y_2 + 3y_3 + y_4 \ge 6$$
$$2y_3 + 3y_4 \ge 10$$
$$y_1, y_2, y_3, y_4 \ge 0$$

Recall standard form for LPs:

Primal LP: $\max\ c^T x$

s.t. $Ax \leq b$

$x \geq 0$

Earlier example:

$$c = \begin{pmatrix} 1 \\ 6 \\ 10 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$b = \begin{pmatrix} 300 \\ 200 \\ 1000 \\ 500 \end{pmatrix}$$

Variables $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Primal LP:

$\max\ c^T x$

s.t. $Ax \leq b$

$x \geq 0$

Dual LP:

$\min\ b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

Variables $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

one variable per constraint

n variables, m constraints $\longrightarrow$ m variables, n constraints

Primal LP:

$$\max\ c^T x \quad \text{s.t. } Ax \le b,\ x \ge 0$$

$\xrightarrow{\text{dual}}$

Dual LP:

$$\min\ b^T y \quad \text{s.t. } A^T y \ge c,\ y \ge 0$$

$\downarrow$ standard form

$$\max\ -b^T y \quad \text{s.t. } -A^T y \le -c,\ y \ge 0$$

$\xleftarrow{\text{dual}}$

$$\min\ -c^T x \quad \text{s.t. } -Ax \ge -b,\ x \ge 0$$

$\uparrow$ standard form

So, dual(dual) = Primal.

For a LP, optimum is achieved at a vertex of the feasible region except if:

a) it's infeasible
(example: $x_1 \le 100$ & $x_1 \ge 200$
so feasible region is empty)

b) it's unbounded
(example: $\max\ x_1 + x_2$
s.t. $x_1, x_2 \ge 0$)

Weak duality theorem:

For a LP whose primal & dual are feasible, then their optimum x^* & y^* satisfy:

$$c^T x^* \le b^T y^*$$

Consequence:

if we have a feasible x of primal LP
& a feasible y of dual LP

and $c^T x = b^T y$

then x & y are optimal solutions.

Strong duality theorem:

If primal LP has optimal x^* & dual LP has optimal y^* then:

$$c^T x^* = b^T y^*$$

If primal LP is unbounded then dual LP is infeasible.
If dual LP is unbounded then primal LP is infeasible.
(could be that both primal & dual LPs are infeasible.)

Another example:

Line fitting:

Given points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
find the line $y = ax + b$ which minimizes the max distance.

Variables: a, b, e

Goal: minimize e where:

for all i, $|y_i - (ax_i + b)| \le e$

y_i: y-coordinate of i^{th} point

$ax_i + b$: y-coordinate of line at x_i

Example: $(x_1, y_1) = (1,1)$, $(x_2, y_2) = (2,1)$, $(x_3, y_3) = (3,2)$

LP: Variables a, b, e

min e

s.t. $|1-a-b| \le e \rightarrow$
$1-a-b \le e$ ①
$1-a-b \ge -e$ ②

$|1-2a-b| \le e \rightarrow$
$1-2a-b \le e$ ③
$1-2a-b \ge -e$ ④

$|2-3a-b| \le e \rightarrow$
$2-3a-b \le e$ ⑤
$2-3a-b \ge -e$ ⑥

In standard form:

$$\max \; -e$$

$$\text{s.t.} \quad -a-b-e \le -1$$
$$a+b-e \le 1$$
$$-2a-b-e \le -1$$
$$2a+b-e \le 1$$
$$-3a-b-e \le -2$$
$$3a+b-e \le 2$$

but how do we get nonnegative constraints?

make variables $a^+, a^-, b^+, b^-, e^+, e^-$

replace $a = a^+ - a^-$
$b = b^+ - b^-$
$e = e^+ - e^-$

New LP: $\max \; -e^+ + e^-$

$$\text{s.t.} \quad -a^+ + a^- - b^+ - b^- - e^+ + e^- \le -1$$
$$a^+ - a^- + b^+ - b^- - e^+ - e^- \le 1$$

$$a^+, a^-, b^+, b^-, e^+, e^- \ge 0$$