Flow network:

directed $G=(V,E)$ with $s \in V$ & $t \in V$
and capacities $c_e > 0$.

A flow is an assignment f_e for each $e \in E$ where:

1) for all $e \in E$, $0 \leq f_e \leq c_e$
2) for all $v \in V - \{s \cup t\}$, $f^{in}(v) = f^{out}(v)$

$$f^{in}(v) = \sum_{\overrightarrow{uv} \in E} f_{uv} \quad \& \quad f^{out}(v) = \sum_{\overrightarrow{vw} \in E} f_{vw}$$

$$\text{Size}(f) = f^{out}(s) = f^{in}(t)$$

Max-flow problem:

Given a flow network find a flow f with $\max \text{size}(f)$.

Closely related is the following min st-cut problem.

What's a cut?

A cut is a partition of the vertices into 2 sets
denote as L & R $(V = L \cup R \ \& \ L \cap R = \emptyset)$

This an st-cut if $s \in L$ & $t \in R$.

For an st-cut define its capacity as:

$$\text{capacity}(L,R) = \sum_{\substack{\vec{vw}\in E: \\ v\in L, w\in R}} c_{vw} = \text{capacity from } L \to R.$$

Min-st-cut problem:

Given a flow network find an st-cut with min capacity.

Max-flow min-cut theorem:

$$\text{size of max-flow} = \text{min capacity of an st-cut}$$

Let's prove the theorem.

The easy direction is that $\text{size of max-flow} \leq \text{min capacity of an st-cut}$.

We'll show that for any flow f & any st-cut (L,R),

$$\text{size}(f) \leq \text{capacity}(L,R) \quad (*)$$

and hence: $\max_f \text{size}(f) \leq \min_{(L,R)} \text{capacity}(L,R)$.

To intuitively see $(*)$ note that f has to cross $L \to R$ & $\leq \text{capacity}(L,R)$ units can cross it.

Let's prove it formally.

For an st-cut (L,R) define

$$f^{in}(L) = \sum_{\substack{\overrightarrow{uv}\in E: \\ u\notin L,\, v\in L}} f_{uv} = \text{flow-into } L$$

$$f^{out}(L) = \sum_{\substack{\overrightarrow{vw}\in E: \\ v\in L,\, w\notin L}} f_{vw} = \text{flow out of } L$$

We'll show that: $\text{size}(f) = f^{out}(L) - f^{in}(L)$.

Why is this true?

$$f^{out}(L) - f^{in}(L) = \sum_{\substack{\overrightarrow{vw}\in E: \\ v\in L,\, w\in R}} f_{vw} - \sum_{\substack{\overrightarrow{uv}\in E: \\ u\in R,\, v\in L}} f_{uv}$$

$$= \sum_{\substack{\overrightarrow{vw}\in E: \\ v\in L,\, w\in R}} f_{vw} - \sum_{\substack{\overrightarrow{uv}\in E: \\ u\in R,\, v\in L}} f_{uv} + \sum_{\substack{\overrightarrow{vw}\in E: \\ v\in L,\, w\in L}} f_{vw} - \sum_{\substack{\overrightarrow{uv}\in E: \\ u\in L,\, v\in L}} f_{uv}$$

these sums are equal just counting from diff. ends.

$$= f^{out}(s) + \sum_{v\in L-\{s\}} \underbrace{\left(f^{out}(v) - f^{in}(v)\right)}_{=0}$$

$$= \text{size}(f) \quad \square$$

Therefore, for any flow f, any st-cut (L,R),

$$\text{size}(f) = f^{out}(L) - f^{in}(L) \leq f^{out}(L) \leq \text{capacity}(L,R).$$

This proves (*).

Now we'll show that: $\max_f \text{size}(f) \geq \min_{(L,R)} \text{capacity}(L,R)$.

Take the flow f^* produced by the Ford-Fulkerson alg.

This flow f^* has no "augmenting path." This means that there is no st-path in the residual network G^{f^*}.

We'll construct an st-cut (L,R) where:

$$\text{size}(f^*) = \text{capacity}(L,R). \qquad (**)$$

Hence, $\max_f \text{size}(f) \geq \min_{(L,R)} \text{capacity}(L,R)$

which is the other half of the max-flow min-cut theorem.

Let's prove (**):

For flow f^*, let L be those vertices reachable from s in G^{f^*}.

We know $t \notin L$ since there is no st-path in G^{f^*}.

Hence let $R = V - L$ & (L,R) is a st-cut.

Consider an edge from $L \to R$:

for $\overrightarrow{vw} \in E$ where $v \in L$ & $w \in R$

we must have $f^*_{vw} = c_{vw}$

otherwise $\overrightarrow{vw}$ in G_{f^*} & then w is reachable from s

Consider an edge from $R \to L$:

for $\overrightarrow{zy} \in E$ where $z \in R$ & $y \in L$:

we must have $f^*_{zy} = 0$

otherwise $\overrightarrow{yz} \in G_{f^*}$ & then z is reachable from s.

Thus, $f^{*out}(L) = \text{capacity}(L,R)$ by

& $f^{*in}(L) = 0$ by

Therefore, $\text{size}(f^*) = f^{*out}(L) - f^{*in}(L)$

$= \text{capacity}(L,R)$

which proves (**).

We also know that any flow f^* with no augmenting paths is a max-flow, since it has $\text{size}(f^*) = \text{min st-cut capacity}$.

An application of max-flow:

Image Segmentation:

Given an image separate it into objects.

Simpler problem: separate it into foreground & background.

Image is on a graph $G=(V,E)$ (undirected) (think of G as a grid)

V = pixels

E = neighboring pixels.

For $i \in V$,

given likelihood a_i that i is in the foreground.

& b_i that i is in the background.

$a_i \geq 0$ & $b_i \geq 0$.

for $(i,j) \in E$,

given separation penalty $P_{ij} \geq 0$.

for partition (A, B) where $V = A \cup B$

A = foreground

B = background

Define its weight as

$$w(A,B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in A: \\ i \in A, j \in B \text{ or} \\ i \in B, j \in A}} p_{ij}$$

Goal: find partition (A, B) with max weight $w(A,B)$.

Reduce to min st-cut problem.

First convert from a maximization to a minimization problem.

& get all signs the same.

Let $Q = \sum_{i \in V} (a_i + b_i)$

Note: $\sum_{i \in A} a_i + \sum_{j \in B} b_j = Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j$

Thus,

$$w(A,B) = Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j - \sum_{\substack{(i,j) \in E: \\ i \in A, j \in B \text{ or} \\ i \in B, j \in A}} p_{ij}$$

Let $w'(A,B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B \text{ or} \\ i \in B, j \in A}} p_{ij}$

(A,B) which maximizes $w(A,B)$

is the same as (A,B) which minimizes $w'(A,B)$

$$w(A,B) = Q - w'(A,B)$$

$$\max_{(A,B)} w(A,B) = Q - \min_{(A,B)} w'(A,B)$$

To reduce to max-flow:

for edge $(i,j) \in E$: add edges $i \to j$ capacity p_{ij}
$j \to i$ capacity p_{ij}

add source s, for every $i \in V$, add edge $s \to i$ capacity a_i

add sink t, for every $j \in V$, add edge $j \to t$ capacity b_j

In this flow network we constructed,

for an st-cut (A,B),

what edges cross $A \to B$?

for $j \in B$ get edge $s \to j$ of capacity a_j

for $i \in A$ get $i \to t$ of capacity b_i

for (i,j) where $i \in A, j \in B$

get $i \to j$ of capacity P_{ij}

& if $i \in B, j \in A$

get $j \to i$ also of capacity P_{ij}.

Thus,

$$\text{Capacity}(A,B) = w'(A,B)$$

So run max-flow, the size of the max-flow

= capacity of min st-cut = $\min_{(A,B)} w'(A,B)$

Then, $\max_{(A,B)} w(A,B) = Q - \min_{(A,B)} w'(A,B)$