Graphics qual questions 2015 spring

GENRAL

Gentle LERP

We want a motion P(t) that starts at point A, so that P(0)=A, with zero velocity, so that it's derivative P'(0)=0, and finishes at B, so that P(1)=B, with zero velocity, so that P'(1)=0, and for which the third derivative is constant, so that P'''[t]=J. What kind of a function is it (a circular arc, a spiral, a parabola…)? What curve constructions popular in geometric design could be used to produce that motion? Hint: Formulate the computation of P(t) in terms of 4 calls to LERP L(C,t,D), which returns C+tCD.

Steady animation of an image

The user picks 3 points in the first frame and specifies their displacement for the second frame. Explain how to compute and render the subsequent frames of a steady animation (where each frame may be computed from the previous one by applying a constant affinity transform T). Discuss advantages of such a motion over other ones and suggest applications.

Free motion of a burning stick

A stick is sliding on the plane without friction or external forces. It has an initial length L, and initial velocity G' of its centroid G, and an initial rotational speed w'. It is burning at both ends with identical and constant speeds, so its total length changes with speed L'. Its mass and thickness are uniform.
(1) How would you animate its motion (for example the position G(t) of its centroid and the orientation w(t) of its direction as a function of time t)?
(2) How would you compute L', G' and w' given two keyframes: its positions, rotations and lengths at times t=0 and t=1.

Hamiltonian split

M is a simple triangle mesh (manifold, zero genus) with v vertices. H is a Hamiltonian cycle of M. Removing H splits the triangles of M into two edge-connected sets: one red and one green. How many triangles are red? Prove your answer. Discuss the answer for the case where M is still manifold, but has genus g.

Counting holes

You are given a **planar** & manifold triangular mesh in the plane. No triangles overlap. The mesh has an unknown number L of manifold bounding loops.

(1) Explain at a high level, how you would compute L by using mesh traversal.

(2) You are told that the mesh has v vertices, t triangles, and b border edges. Can you compute L directly without traversing the mesh? It not, explain why not. If yes, explain how (provide the formula) and justify your answer formally.

J-splines

Are J-spline affine invariant? Explain the question. Suggest why it is important in graphics. Provide your answer and justify it with a counter-example, proof, or argument.

MODELING

Bouncing ball

You are given a connected, manifold polyhedron (solid) P. We want to simulate the bouncing motion of a ball B of radius r inside P.

(1) Provide a succinct and elegant mathematical expression of a sufficient and necessary condition on P that guarantees that B will never hit an edge or vertex of P.

(2) Propose, at a high level, a practical algorithm for testing this condition.

(3) Provide and justify the computational complexity of your algorithm.

Hint 1: P does not need to be convex.

Hint 2: In your formulation, you may use $P_r = !((!P)^r)$, where P^r is the set of points at distance r or less from P and assume that you are given an algorithm for computing it.

Topology change

You are given an array of points Pi and one-by-one a list of triangles, each defined by the 3 indices of its vertices You are guaranteed that the relative interiors of the triangles are pairwise disjoint. But more then two triangles can be incident upon the same edge. Explain how you would implement an algorithm that identifies the first triangle, the addition of which disconnects space (i.e., closes the door between the infinite exterior and some interior room).

Keyframed Minkowski morph
You are given 3 convex polyhedra, A, B, C, each bounded by a triangle mesh and are told that the set of all their vertices is in general configuration (no 4 vertices are coplanar).
Explain how to produce a smooth animation of a polyhedron P(t), such that P(-1)=A, P(0)=B, and P(1)=C, using a weighted Minkowski average P(t)=a(t)A+b(t)B+c(t)C.
(1) Provide a clear and concise definition of the weighted Minkowski average of 3 arbitrary sets.
(2) Provide the weight functions a(t), b(t), and c(t).
(3) Explain which property that holds for convex sets you will use in your construction.
(4) Provide a clear and precise description of how you would implement show(a(t),A,b(t),B,c(t),C) that display P(t) directly from the input meshes and weights, one face at a time, without using any complicated data structure.

Counting rooms in a bubble house
What is the maximum number c(n) of full-dimensional cells in an arrangement of n spheres in space? If you cannot crack the formula, at least provide its O complexity and a detailed justification.

Curve similarity
You are given two closed-loop curves C0 and C1 in 3D. Each one is a PCC (Piecewise-Circular Curve) made of smoothly connected circular arcs. Explain how you would compute their Hausdorff distance between them. Hint: You may attempt a mathematically correct approach or a simple to implement iterative scheme that computes an accurate approximation. Provide a high level outline of your solution first, then identify the delicate or challenging modules, and explain how you would address then (you may not have the time to develop the details).

RENDERING

Reflection of a reflection

There is a window with vertices A, B, C, D and a circular puddle on the floor with center G and radius r.

Your eye is at E and the lamp (point source) is at L. Provide the details of implementing a test to establish whether you see in the puddle the reflection of the window in which you see the reflection of the lamp (i.e., the ray from the lamp bounces off the window, then off the puddle, and reaches your eye exactly. We assume perfect mirror reflections. Provide the fill details of the geometric test using points, vectors, and operations. Be complete, concise, and clear. Justify your approach and explain each derivation.

FOR THE ORAL

We want a motion P(t) that starts at point A [P(0)=A] with zero velocity [P'(0)=0] and finishes at B [P(1)=B] with zero velocity [P'(1)=0] and for which the third derivative is constant [P'''[t]=J]. What kind of a function is it (a circular arc, a spiral, a parabola…)? What curve constructions popular in geometric design could be used to feeding that motion?

Formulate the computation of P(t) in terms of 4 calls to LERP , using the notation L(C,t,D) which returns C+tCD.

Cubic polynomial, cubic Hermite which may be converted to a Cubic Bezier, L(L(A,t,L(A,t,B)),t,L(L(A,t,B),t,B))

Define the lowest degree polynomial expression for a trajectory P(t) of a particle that satisfies the following constraints, where $P'(t)$ denotes its velocity vector at time t: $P(-1)=A$, $P'(-1)=U$, $P(0)=B$, $P'(1)=V$, and $P(1)=C$. You may provide either the polynomial expression, such as $P(t) = A + tB + t^2V\ldots$, or a construction algorithm, such as for $P(t)$.

$P(t) = t^4X + t^3Y + t^2Z + tW + O$
$O=B$
$BA=X-Y+Z-W$
$BC=X+Y+Z+W$
$2(X+Z)=BA+BC$
$2(W+Y)=AC$

$P'(t) = 4t^3X + 3t^2Y + 2tZ + W$
$U = -4X + 3Y - 2Z + W$
$V = 4X + 3Y + 2Z + W$
$U+V = 6Y + 2W$
$V-U = 8X + 4Z$

$2X + 2Z = BA + BC$
$8X + 4Z = V - U$

$V - U - 2BA - 2BC = 4X$
$4BA+4BC-V+U = 4Z$

$6Y + 2W = U + V$
$2Y + 2W = AC$

$U+V-AC = 4Y$

$3AC-U-V = 4W$

$X = (V - U - 2BA - 2BC)/4$
$Y = (U+V-AC)/4$
$Z = (4BA+4BC-V+U)/4$
$W = (3AC-U-V)/4$
$O=B$

What is the maximum number c(n) of full-dimensional cells in an arrangement of n spheres in space.
c(n+1):
surface of new sphere is cut by at most n circles.
Each circle is split into at most 2+2(n-2) = 2n-2 = 2(n-1) arcs by vertices where it intersects other spheres.
so
c(n+1)=c(n)+2n(n-1)
c(n)=c(n-1)+2(n-1)(n-1)
c(1)=1+1
c(2)=1+1+2=3
c(3)=1+1+2*1^2+2*2^2
c(4)=1+1+2*1^2+2*2^2+2*3^2
C(n) = 2(1+1^2+2^2+…(n-1)^2) = 2(n^3/3+n^2/2+n/6)
http://www.trans4mind.com/personal_development/mathematics/series/sumNaturalSquares.htm

http://math.stackexchange.com/questions/1146128/scale-position-points-in-a-circle-look-like-normal-scaling

Concentricity test

http://math.stackexchange.com/questions/1143207/how-to-prove-that-n-points-lie-on-a-cirlce

Cut

Let R be a simply connected planar region bounded by a smooth piecewise circular Jordan curve B. Suggest a reasonably efficient algorithm for computing the exact shape of a minimum length cut C that will split R in two regions of equal area.

Provide explicit construction for the point M that minimizes the sum of the distances to points A, B, and C

In general, a point that minimizes the sum of distances is the geometry median, also known as L1 estimator or the Torricelli point.

For 3 points, M is the Fermat point of the triangle and under some configurations, lines joining M to A, B, and C form a minimal Steiner tree.

http://en.wikipedia.org/wiki/Geometric_median
http://en.wikipedia.org/wiki/Fermat_point

Transfer texture between 2 convex quads (what to optimize and how to implement it)
http://www.reedbeta.com/blog/2012/05/26/quadrilateral-interpolation-part-1/
http://math.stackexchange.com/questions/13404/mapping-irregular-quadrilateral-to-a-rectangle
http://www.fho-emden.de/~hoffmann/persprect13052005.pdf

Point that minimizes the sum of the distances to 2 skewed lines in space