Curves and Surfaces

Curves

- Implicit/Explicit
- Piecewise
- Parametric cubic curves
  - Basic ideas
  - Hermite
  - Bezier
  - B-Splines
Explicit/Implicit

- **Explicit Functions:** \( y = f(x) \) (e.g., \( y = 2x \))
  - Only one value of \( y \) for each \( x \)
  - Not rotationally invariant
  - Difficult to represent a slope of infinity

- **Implicit Equations:** \( f(x,y) = 0 \) (e.g., \( x^2 + y^2 - r^2 = 0 \))
  - Need constraints to model part of a curve
  - Joining curves together smoothly is difficult

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Piecewise Parametric Curves

- **Piecewise curves**
  - Use multiple simple curves to model a complex curve in pieces

- **Parametric Equations:**
  for \( 0 \leq t \leq 1, \ x = f(t), \ y = g(t) \) (e.g., \( x = 3 + 4t, \ y = 2 - 2t \))
  - Easy to join curve segments smoothly
  - Slopes are parametric tangent vectors
Piecewise Curves: Continuity

- $G^0$: geometric continuity
  - Curve segments join together
- $G^1$: geometric continuity
  - Tangent vectors equal directions at join point
- $C^1$: parametric continuity
  - Tangents have equal magnitude and direction
- $C^2$: parametric continuity
  - Direction and magnitude of $n^{th}$ derivative equal

Continuity Examples
Parametric Cubic Curves

- Use cubic curves
  \[ x = a_xt^3 + b_xt^2 + c_xt + d_x \]
  \[ y = a_yt^3 + b_yt^2 + c_yt + d_y \]
  \[ z = a_zt^3 + b_zt^2 + c_zt + d_z \]

- Notation: \( Q(t) = T \mathbf{C} \), where
  \[
  T = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix}
  \]
  \[
  \mathbf{C} = \begin{pmatrix}
  a_x & a_y & a_z \\
  b_x & b_y & b_z \\
  c_x & c_y & c_z \\
  d_x & d_y & d_z 
  \end{pmatrix}
  \]

Why Cubic?

- Lower: inflexible
- Higher: hard to control, expensive

- 4 coefficients \( \leftrightarrow \) 4 unknowns needed
  - e.g. endpoints, tangents, continuity
  - What they are determine the kind of curve
Types of Curves

- Hermite
  - 2 endpoints, 2 endpoint tangents
- Bezier
  - 2 endpoints, 2 other points defining tangents
- B-Splines
  - 4+ control points, \( C^0 \) continuous
  - approximates control points

Types of Curves

- Non-Uniform B-Splines
  - Control points (knots) can be repeated
    - Curve can be forced through control points
    - Sharp corners can be created
- Rational curves
  - 3D curves modeled in 4-space
    - Control points are \((xw, yw, zw, w)\)
  - Weight can pull curve toward control points
  - Non-affine transforms (e.g., projection)
Defining Cubic Polys

\[ Q(t) = TC = TMG \]

\[ = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix} \]

T Matrix    Basis Matrix    Geometry Matrix

Defining Cubic Polys

i.e. \( x(t) = TMG_x \)

\[ = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} g_{1x} \\ g_{2x} \\ g_{3x} \\ g_{4x} \end{pmatrix} \]

Similarly for \( y(t) \) and \( z(t) \)
Defining Cubic Polys

- Basis matrix defines type of cubic
- \( \mathbf{TM} \) \( \rightarrow \) 4 cubic polynomials
  - *blending functions*
  - Defined to achieve desired props for \( G \)
- Curve is weighted sum of elements of \( G \)
  - Weights are cubics in \( t \)

2D example

- Recall \( \mathbf{Q}(t) = \mathbf{T} \mathbf{M} \mathbf{G} \)
- Lines:
  - \( \mathbf{x}(t) = (1-t) \mathbf{g}_{1x} + (t) \mathbf{g}_{2x} \)
- Blending functions
  - \(-t + 1, \ t\)
- \( \mathbf{M} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \)
Aside: Derivatives

- Parametric Curves
  \[ x = a_xt^3 + b_xt^2 + c_xt + d_x \]
  \[ y = a_yt^3 + b_yt^2 + c_yt + d_y \]
  \[ z = a_zt^3 + b_zt^2 + c_zt + d_z \]

- Derivatives of Parametric Curves
  \[ \frac{dx}{dt} = 3a_xt^2 + 2b_xt + c_x \]
  \[ \frac{dy}{dt} = 3a_yt^2 + 2b_yt + c_y \]
  \[ \frac{dz}{dt} = 3a_zt^2 + 2b_zt + c_z \]

Hermite Curves

- \( Q(t) = T M_H G_H \)
  - \( M_H \) is the Hermite Basis Matrix
  - \( G_H \) is the Hermite Geometry Matrix
- \( G_H \): 4 triples that define the curve
  - 2 endpoints \((P_1 \ P_4)\)
  - 2 endpoint tangents \((R_1 \ R_4)\)
- What is \( M_H \)?
Hermite Basis Matrix $M_H$

- Given: $P_1, P_4$ and $R_1, R_4$
- Recall: $x(t) = TM_HG_{Hx}, \ x'(t) = T'M_HG_{Hx}$
- Thus: $x(0) = P_{1x} = [0 \ 0 \ 0 \ 1] M_HG_{Hx}$
  $x(1) = P_{4x} = [1 \ 1 \ 1 \ 1] M_HG_{Hx}$
  $x'(0) = R_{1x} = [0 \ 0 \ 1 \ 0] M_HG_{Hx}$
  $x'(1) = R_{4x} = [3 \ 2 \ 1 \ 0] M_HG_{Hx}$

Hermite Basis Matrix

- But,
  $[P_{1x}] \ [0 \ 0 \ 0 \ 1]$
  $[P_{4x}] = G_{Hx} = [1 \ 1 \ 1 \ 1] M_HG_{Hx}$
  $[R_{1x}] \ [0 \ 0 \ 1 \ 0]$
  $[R_{4x}] = [3 \ 2 \ 1 \ 0]$
- Thus, $M_H$ = inverse of above matrix
  $[2 \ -2 \ 1 \ 1]$
  $= [-3 \ 3 \ -2 \ 1]$
  $[0 \ 0 \ 1 \ 0]$
  $[1 \ 0 \ 0 \ 0]$
Hermite Blending Functions

\[ Q(t) = T M_h G_h \]
\[ = (2t^3 - 3t^2 + 1)P_1 + \]
\[ (-2t^3 + 3t^2)P_4 + \]
\[ (t^3 - 2t^2 + t)R_1 + \]
\[ (t^3 - t^2)R_4 \]

Bezzer Curves

\[ Q(t) = T M_B G_B \]
\[ M_B \] is the Bezier Basis Matrix
\[ G_B \] is the Bezier Geometry Matrix

\[ G_B \]: 4 control points
\[ 2 \] endpoints \((P_1, P_4)\)
\[ 2 \] points defining tangents \((P_2, P_3)\), where
\[ R_1 = 3(P_2 - P_1) \]
\[ R_4 = 3(P_4 - P_3) \]
Beziers Curves

- Bounded by convex hull of control points

\[
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4
\end{bmatrix}
\]

Beziers Basis Matrix

- But,

\[
G_B = \begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4
\end{bmatrix},
\quad
G_H = \begin{bmatrix}
P_1 \\
P_4
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix},
\quad
G_B = M_{HB}G_B
\]

\[
\begin{bmatrix}
P_2 \\
R_1 \\
P_4 \\
R_4
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1
\end{bmatrix},
\quad
G_B = M_{HB}G_B
\]

- Thus, \( M_B = M_HM_{HB} \)

\[
\begin{bmatrix}
-1 & 3 & -3 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3 & -6 & 3 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-3 & 3 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}
\]
Bezier Blending Functions

- The Bernstein Polynomials
- \( Q(t) = T M_b G_b \)
  \[
  = (1-t)^3 P_1 + \\
  3t(1-t)^2 P_2 + \\
  3t^2(1-t) P_3 + \\
  t^3 P_4 
  \]