4.3 For μ in the range $[1, \ln n]$, use (4.1) to obtain a closed-form upper bound for $\Delta^+(\mu, 1/n^2)$ (as a function of μ and n) that is within a constant factor of the best possible.

4.4 Let $X_1, X_2, \ldots, X_n$ be independent geometrically distributed random variables each having expectation 2 (each of the X_i is an independent experiment counting the number of tosses of an unbiased coin up to and including the first HEADS). Let $X = \sum_{i=1}^n X_i$ and δ be a positive real constant. Use moment generating functions and the Chernoff technique to derive the best upper bound you can on $\mathbf{Pr}[X > (1+\delta)(2n)]$.

4.5 The result of Theorem 4.2 bounds the probability of the sum of Poisson trials deviating far *below* its expectation. Use this to give a bound on the probability of the sum of independent geometric random variables deviating *above* its expectation, thus providing an alternative approach to that in Problem 4.4.

4.6 (**Hoeffding's Bound [202]**). Suppose $Y_1, \ldots, Y_n$ are independent Poisson trials such that $\mathbf{Pr}[Y_i = 1] = p_i$. Let $Y = \sum_{i=1}^n Y_i$, $\mu = \mathbf{E}[Y] = \sum_{i=1}^n p_i$ and $p = \mu/n$. Our goal is to show that from the standpoint of deviations from the mean, the worst case is when the p_i's are all equal. Let X be the sum of n independent Bernoulli trials each having probability p of assuming the value 1. Then, for any $a \geq \mu + 1$ and any $b \leq \mu - 1$, show that

$$\mathbf{Pr}[Y \geq a] \leq \mathbf{Pr}[X \geq a],$$

and

$$\mathbf{Pr}[Y \leq b] \leq \mathbf{Pr}[X \leq b].$$

4.7 (Due to W. Hoeffding [202].) This problem deals with a useful generalization of the Hoeffding bound in Problem 4.6.

(a) A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *convex* if for any x_1, x_2 and $0 \leq \lambda \leq 1$, the following inequality is satisfied:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

Show that the function $f(x) = e^{tx}$ is convex for any $t > 0$. What can you say when $t \leq 0$?

(b) Let Z be a random variable that assumes values in the interval $[0, 1]$, and let $p = \mathbf{E}[Z]$. Define the Bernoulli random variable X such that $\mathbf{Pr}[X = 1] = p$ and $\mathbf{Pr}[X = 0] = 1 - p$. Show that for any convex function f, $\mathbf{E}[f(Z)] \leq \mathbf{E}[f(X)]$.

(c) Let $Y_1, \ldots, Y_n$ be independent and identically distributed random variables over $[0, 1]$, and define $Y = \sum_{i=1}^n Y_i$. Using parts (a) and (b), derive upper and lower tail bounds for the random variable Y using the Chernoff bound technique. In particular, show that

$$\mathbf{Pr}[Y - \mathbf{E}[Y] > \delta] \leq \exp(-2\delta^2/n).$$

Remark: While the results in this problem hold for continuous random variables, they may be a bit easier to prove in the case where $Z, Y_1, \ldots, Y_n$ take on a discrete set of values in the interval $[0, 1]$. Also, it should be easy to generalize this to distributions defined over arbitrary intervals $[l, h]$. See also Problem 4.21.