Constrained rigid body
• Collision detection

• Contact point

• Colliding contact

• Resting contact

• Friction
Collision detection

- Determine whether the collision occurs within a numerical tolerance
- Determine all pairs of bounding boxes that overlap
- Further check contact points between rigid bodies defined as convex polyhedra
Bisection

\[ Y(t_0) \]

\[ Y(t_0 + \frac{1}{2} \Delta t) \]

\[ Y(t_c) \]

\[ Y(t_0 + \Delta t) \]
Bounding boxes

• Use bounding boxes to reduce the number of pairwise collision/contact determinations

• If two bounding boxes have no overlap, no further comparisons are needed

• Coherence can substantially improve the performance
Bounding boxes

Sweep and sort algorithm

\[ I_1, I_2, I_3, I_4, I_5, I_6 \]
Bounding boxes

- Each bounding box $i$ can be described as an interval $(b_i, e_i)$
- Find all pairs of interval that intersect
  - All the $b_i$ and $e_i$ are sorted from lowest to highest in a list
  - Maintain an active list which is initially empty
  - Whenever $b_i$ is encountered, all intervals on the active list are output as overlapping with $i$, and $i$ is added to the list
Bounding boxes

Sweep and sort algorithm
• We only need to do a full sort and sweep initially
  • What is the cost of the sorting process?
• Subsequent comparisons can be improved by utilizing coherence
  • What is the best sorting algorithm for a nearly sorted list?
Convex polyhedra

Two polyhedra do not inter-penetrate if and only if a separating plane between them exists.
Check all possible edges or faces to find a separating plane at the very first time step or the first time the bounding boxes overlap.
Convex polyhedra

Utilize coherence by caching defining faces or defining edges

$Y(t_0)$

$Y(t_0 + \Delta t)$
Convex polyhedra

The defining face no longer separate the polyhedra and a new separating plane must be found.

If no separating plane is found, an inter-penetration has occurred at some earlier time.

The simulator must back up until the collision time is determined.
• Collision detection
• Contact point
• Colliding contact
• Resting contact
• Friction
Contact points

- Once we determine which bodies contact, we also need to determine the exact contact points
- We consider two types of contacts
  - vertex/faces
  - edge/edge
Contact points

- 4 vertex/face
- 2 vertex/face
- 2 edge/edge
Although $p_a(t)$ and $p_b(t)$ are coincident at time $t_c$, the velocity of the two points may be different.
Velocity of the contact point

This quantity gives the component of the relative velocity in the normal direction
Relative normal velocity

\[ v_r > 0 \]

Separation

\[ v_r = 0 \]

Resting contact

\[ v_r < 0 \]

Colliding contact
• Collision detection
• Contact point
• Colliding contact
• Resting contact
• Friction
Collision process

Impulse: $\int f \, dt = J = M \Delta v$
A soft collision

Force

$\Delta t$

Velocity

$\Delta V$
A harder collision

Force

\[ \Delta t \]

Velocity

\[ \Delta V \]
An infinitely hard collision

\[ \Delta t = 0 \]

\[ f_{imp} = \infty \]

\[ f_{imp} \Delta t = J \]
• In the rigid body world, we want velocity to change instantaneously
  • use finite impulse to change velocity instead of infinite force
  • \( J = \Delta P = M \Delta v \)
Impulsive torque

• If the impulse acts on a point \( p \), \( J \) produces an impulsive torque

\[ \tau_{imp} = (p - x(t)) \times J \]

• Impulsive torque gives rise to a change in angular momentum: \( \tau_{imp} = \Delta L = I(t_c) \omega(t_c) \)
Colliding contact

When two bodies collide, we apply an impulse between them to change their velocity.

For frictionless bodies, the direction of the impulse will be in the normal direction \( \hat{n}(t_c) \)

\[
\mathbf{J} = j\hat{n}(t_c)
\]

Body A is subject to this impulse \( \mathbf{J} \), while body B is subject to an equal but opposite impulse \(-\mathbf{J}\)
Colliding contact

Relative normal velocity before and after the application of the impulse

\[ v_r^- = \hat{n}(t_c) \cdot (\dot{p}_a^-(t_0) - \dot{p}_b^-(t_0)) \]
\[ v_r^+ = \hat{n}(t_c) \cdot (\dot{p}_a^+(t_0) - \dot{p}_b^+(t_0)) \]

To solve for \( j \), we need one more piece of information.
The empirical law

Coefficient of restitution:

\[ v_r^+ = -\epsilon v_r^- \]

\( \epsilon = 0 \) Resting contact

\( \epsilon = 1 \) Perfect bounce
Compute the impulse

Now we are ready so solve for the impulse

Define the displacements from center of mass of the body

\[ r_a = p_a(t_c) - x_a(t_c) \quad \text{and} \quad r_b = p_b(t_c) - x_b(t_c) \]

The pre-impulse velocity of \( p_a \): \[ \dot{p}_a^- = v_a^- (t_c) + \omega_a^- (t_c) \times r_a \]

The post-impulse velocity of \( p_a \): \[ \dot{p}_a^+ = v_a^+ (t_c) + \omega_a^+ (t_c) \times r_a \]

Replace \( v_a^+ (t_c) \) with \( v_a^+ (t_c) = v_a^- (t_c) + \frac{j \hat{n}(t_c)}{M_a} \) in \( \dot{p}_a^+ (t_c) \)

Replace \( \omega_a^+ (t_c) \) with \( \omega_a^+ (t_c) = \omega_a^- (t_c) + I^{-1}(t_c)(r_a \times j \hat{n}(t_c)) \) in \( \dot{p}_a^+ (t_c) \)
Compute the impulse

\[ \dot{p}_a^+(t_c) = \dot{p}_a^- + j \left( \frac{\hat{n}(t_c)}{M_a} + (I_a^{-1}(t_c)(r_a \times \hat{n}(t_c))) \times r_a \right) \]

\[ \dot{p}_b^+(t_c) = \dot{p}_b^- - j \left( \frac{\hat{n}(t_c)}{M_b} + (I_b^{-1}(t_c)(r_b \times \hat{n}(t_c))) \times r_b \right) \]

\[ v_r^+ = \hat{n}(t_c) \cdot (\dot{p}_a^+(t_c) - \dot{p}_b^+(t_c)) \]

\[ = \hat{n}(t_c) \cdot (\dot{p}_a^-(t_c) - \dot{p}_b^-(t_c)) + j \left( \frac{1}{M_a} + \frac{1}{M_b} + \hat{n}(t_c) \cdot (I_a^{-1}(t_c)(r_a \times \hat{n}(t_c))) \times r_a + \hat{n}(t_c) \cdot (I_b^{-1}(t_c)(r_b \times \hat{n}(t_c))) \times r_b \right) \]

\[ = v_r^- + j \left( \frac{1}{M_a} + \frac{1}{M_b} + \hat{n}(t_c) \cdot (I_a^{-1}(t_c)(r_a \times \hat{n}(t_c))) \times r_a + \hat{n}(t_c) \cdot (I_b^{-1}(t_c)(r_b \times \hat{n}(t_c))) \times r_b \right) \]

\[ v_r^- + j \left( \frac{1}{M_a} + \frac{1}{M_b} + \hat{n}(t_c) \cdot (I_a^{-1}(t_c)(r_a \times \hat{n}(t_c))) \times r_a + \hat{n}(t_c) \cdot (I_b^{-1}(t_c)(r_b \times \hat{n}(t_c))) \times r_b \right) = -\epsilon v_r^- \]
Compute the impulse

\[ j = \frac{1}{M_a} + \frac{1}{M_b} + \hat{n}(t_c) \cdot (I^{-1}_a(t_c)(r_a \times \hat{n}(t_c))) \times r_a + \hat{n}(t_c) \cdot (I^{-1}_b(t_c)(r_b \times \hat{n}(t_c))) \times r_b \]

Now we can compute the impulse and impulsive torque for body A and B

A: \( J = j \hat{n}(t_c) \)
\( \tau_{imp} = r_a \times J \)

B: \( J = -j \hat{n}(t_c) \)
\( \tau_{imp} = r_b \times J \)

Finally, apply the change in linear momentum and angular momentum to the current state

\[ P(t_c + h) = P(t_c) + J \]
\[ L(t_c + h) = L(t_c) + \tau_{imp} \]
• Collision detection
• Contact point
• Colliding contact
• Resting contact
• Friction
Resting contact

• In this case, all $n$ contact points have the zero relative velocity

• At each contact point there is some force $f_i \hat{n}(t_c)$, where $f_i$ is an unknown scalar

• Our goal is to determine what each $f_i$ is and all the $f_i$'s must be determined at the same time
Resting contact

To solve for $f_i$, we need to consider

- three relations between acceleration of the contact points and the contact forces
- all the $f_i$ that are in contact at the same time
Non-penetration

\[ d_i(t) = \hat{n}_i(t) \cdot (p_a(t) - p_b(t)) \]

\[ d_i(t) > 0 \]

\[ d_i(t) = 0 \]

\[ d_i(t) < 0 \]

This is what we want to avoid
Non-penetration

\[ \dot{d}_i(t) = \hat{n}_i(t) \cdot (p_a(t) - p_b(t)) + \hat{n}_i \cdot (\dot{p}_a(t) - \dot{p}_b(t)) \]

At time \( t_c \), \( p_a(t_c) = p_b(t_c) \)  
This means that

\[ \dot{d}_i(t_c) = v_r \]

The definition of two bodies in resting contact:

\[ d_i(t_c) = \dot{d}_i(t_c) = 0 \]
Non-penetration

When \( d_i(t_c) = \dot{d}_i(t_c) = 0 \)

If \( \ddot{d}_i(t_c) < 0 \), the bodies have an acceleration towards each other and the penetration will occur.

Therefore, the first condition is

\[ \ddot{d}_i(t_c) \geq 0 \]
The contact forces can push bodies apart, but can never act like “glue” and hold bodies together.

Therefore, each contact force must act outward.

\[ f_i \geq 0 \]
The contact force at the contact point becomes zero if the bodies begin to separate.

If contact is breaking \( \ddot{d}_i(t_c) > 0 \), then \( f_i \) should be zero.

If \( f_i \) is not zero, then the contact is not breaking \( \ddot{d}_i(t_c) = 0 \).

What is the equation that satisfies these two cases?

\[ f_i \ddot{d}_i(t_c) = 0 \]
Resting contact conditions

Condition 1: Non-penetration

\[ \ddot{d}_i(t_c) \geq 0 \]

where

\[ d_i(t) = \hat{n}_i(t) \cdot (p_a(t) - p_b(t)) \]

Condition 2: Repulsive force

\[ f_i \geq 0 \]

Condition 3: Workless force

\[ f_i \ddot{d}_i(t_c) = 0 \]
Compute contact forces

\[
\ddot{d}_i(t_c) = a_{i1} f_1 + a_{i2} f_2 + \cdots + a_{in} f_n + b_i
\]

\[
\begin{bmatrix}
\ddot{d}_1(t_c) \\
\vdots \\
\ddot{d}_n(t_c)
\end{bmatrix}
= A \begin{bmatrix}
f_1 \\
\vdots \\
f_n
\end{bmatrix}
+ b
\]

Step 1: compute \( A \) and \( b \)

Step 2: solve \( f \) using quadratic programming
Compute contact forces

\[
\ddot{d}_i(t_c) = \hat{n}_i(t_c) \cdot (\dot{p}_a(t_c) - \dot{p}_b(t_c)) + 2\hat{n}_i(t_c) \cdot (\dot{p}_a(t_c) - \dot{p}_b(t_c))
\]

\[
= a_{i1} f_1 + a_{i2} f_2 + \ldots + a_{in} f_n + b_i
\]

Factor out the terms that depend on \(f_j\) and assign them to \(a_{ij}\)

Assign the rest of terms to \(b_i\)

See details in Baraff and Witkin’s course notes
Quadratic programming

Solve for $f_1, f_2, \cdots, f_n$

Subject to

$$A \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} + b \geq 0$$

$$f_i \geq 0 \quad i = 1 \cdots n$$

$$(a_{i1}f_1 + \cdots + a_{in}f_n + b_i)f_i = 0 \quad i = 1 \cdots n$$
• Collision detection
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Contact friction

- Dry friction occurs when surfaces in contact are free of any contamination.
- Wet friction occurs when the surfaces are separated by a thin film of lubricants.
Dry contact friction

- Coulomb’s law

- if sliding, the kinetic friction is

\[ \mathbf{F}_{\text{friction}} = -\mu_k |\mathbf{F}_{\text{normal}}| \frac{\mathbf{v}_T}{|\mathbf{v}_T|} \]

- if static (\(\mathbf{v}_T = 0\)) then stay static as long as

\[ |\mathbf{F}_{\text{friction}}| \leq \mu_s |\mathbf{F}_{\text{normal}}| \]
Given $\mu$, can you compute $\theta$?
Summary

- What’s the physical meaning of the rotation matrix?
- How is the rotation matrix related to angular velocity?
- How do you compute inertia tensor efficiently?
- Why do we compute impulse instead of force in the case of colliding contact?