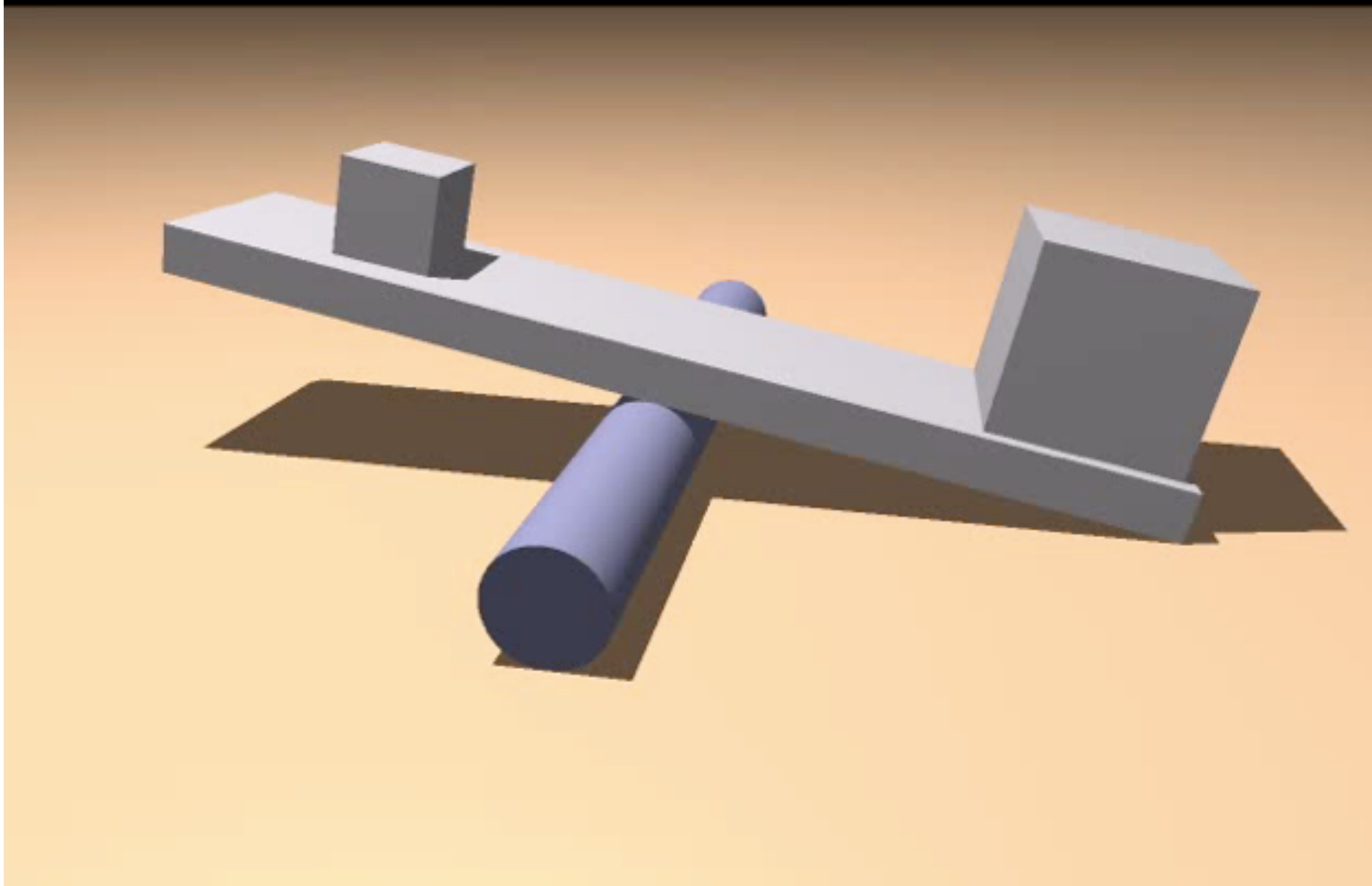
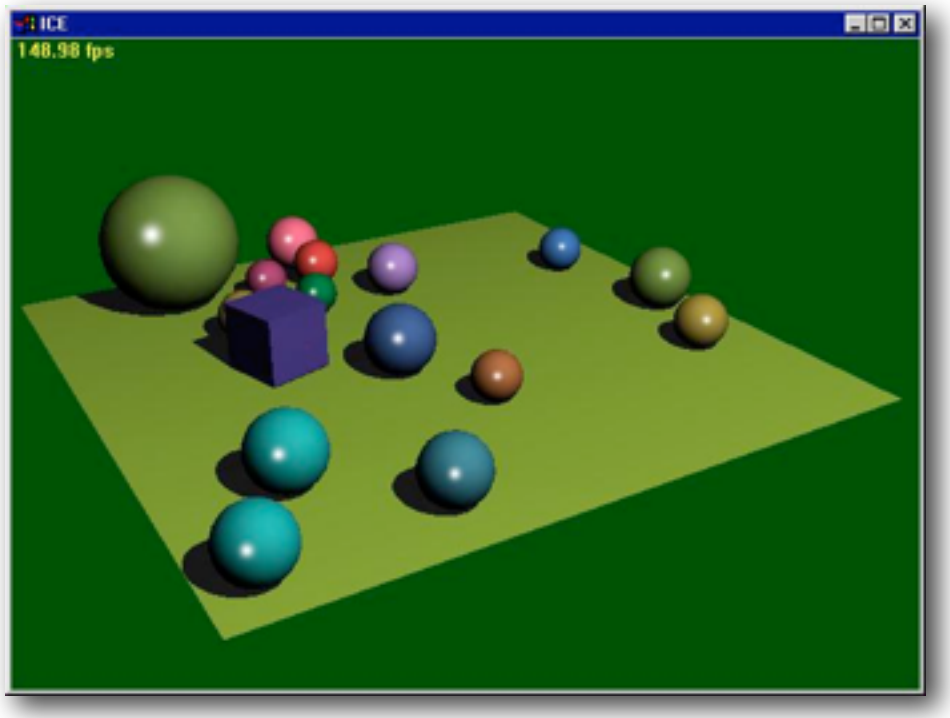
Rigid body dynamics

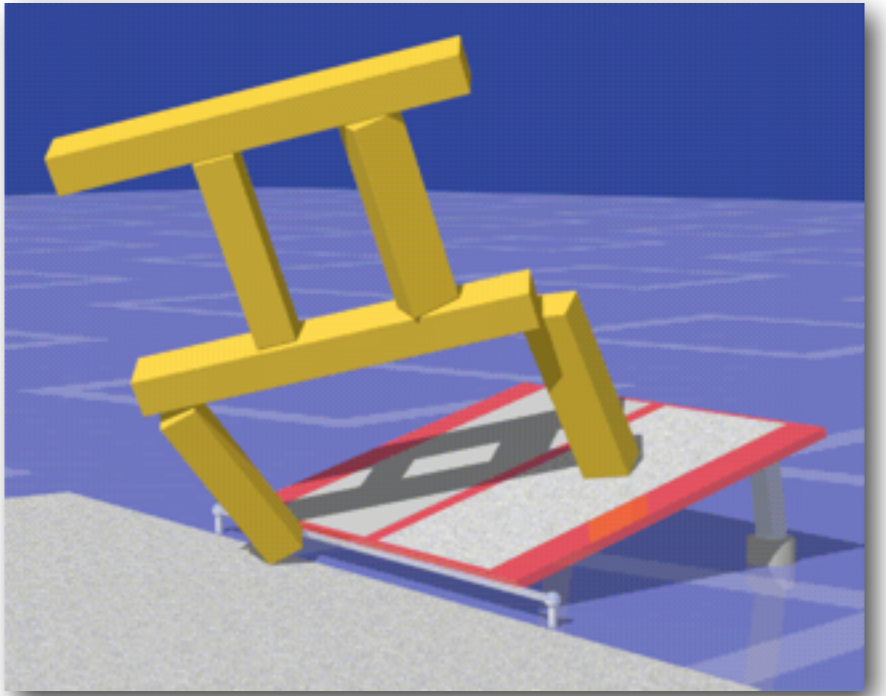


Rigid body simulation



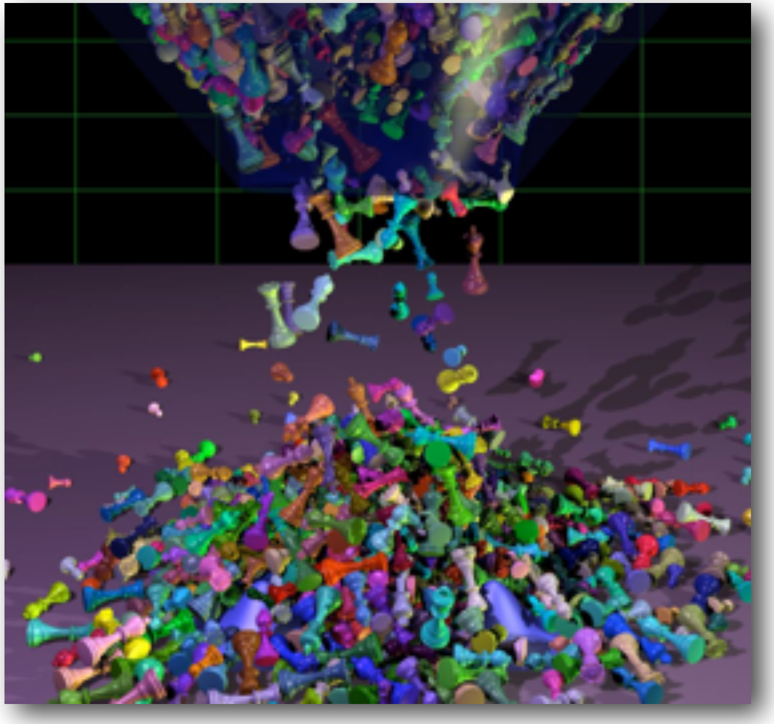
Once we consider an object with spatial extent, particle system simulation is no longer sufficient

Rigid body simulation



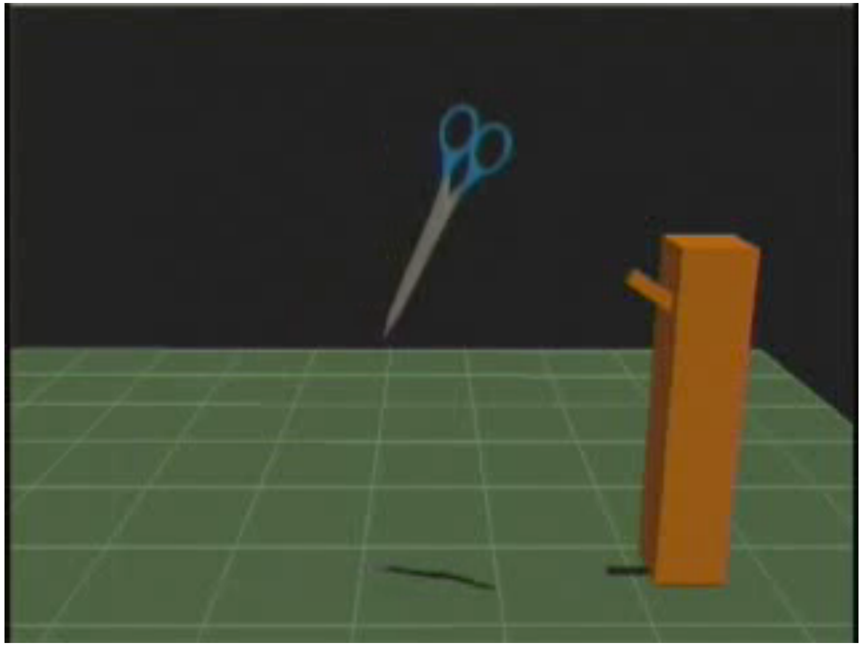
- Unconstrained system
  - no contact
- Constrained system
  - collision and contact

Problems



Performance is important!

Problems



Control is difficult!

Particle simulation

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{bmatrix}$$

Position in phase space

$$\dot{\mathbf{Y}}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{f}(t)/m \end{bmatrix}$$

Velocity in phase space

Rigid body concepts

Translation

Position
Linear velocity
Mass
Linear momentum
Force

Rotation

Orientation
Angular velocity
Inertia tensor
Angular momentum
Torque

- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

Position and orientation

Translation of the body

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotation of the body

$$\mathbf{R}(t) = \begin{bmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{xz} & r_{yz} & r_{zz} \end{bmatrix}$$

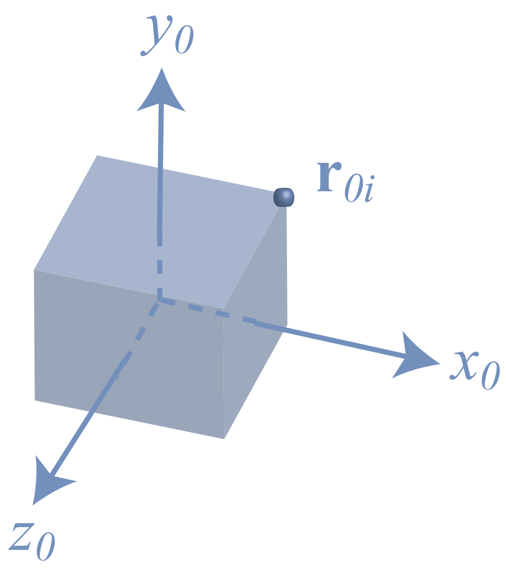
$\mathbf{x}(t)$ and $\mathbf{R}(t)$ are called *spatial variables* of a rigid body

Quiz

- True or False: Given an arbitrary rotation matrix R
  - R is always orthonormal
  - R is always symmetric
  - $RR^T = I$
  - $R_x(30)R_y(60) = R_y(60)R_x(30)$

Body space

Body space

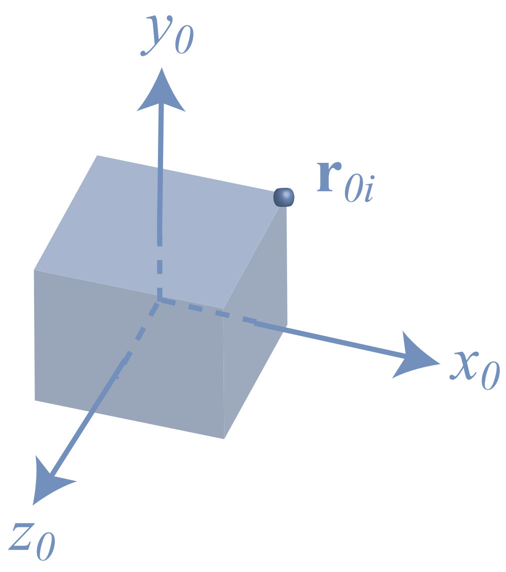


A fixed and unchanged space where the shape of a rigid body is defined

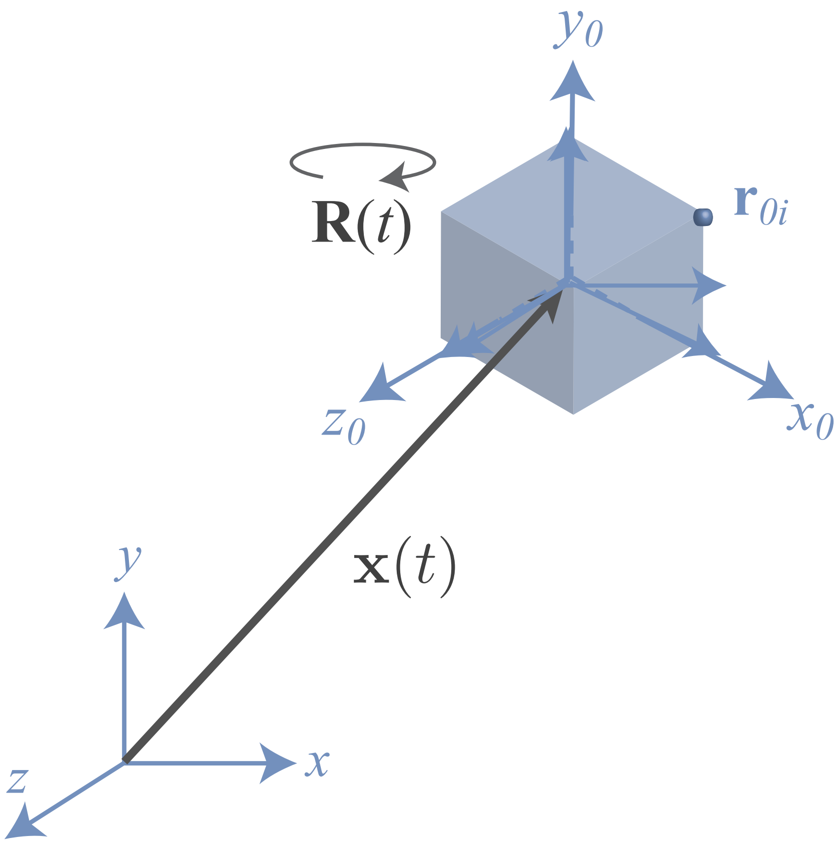
The geometric center of the rigid body lies at the origin of the body space

Position and orientation

Body space



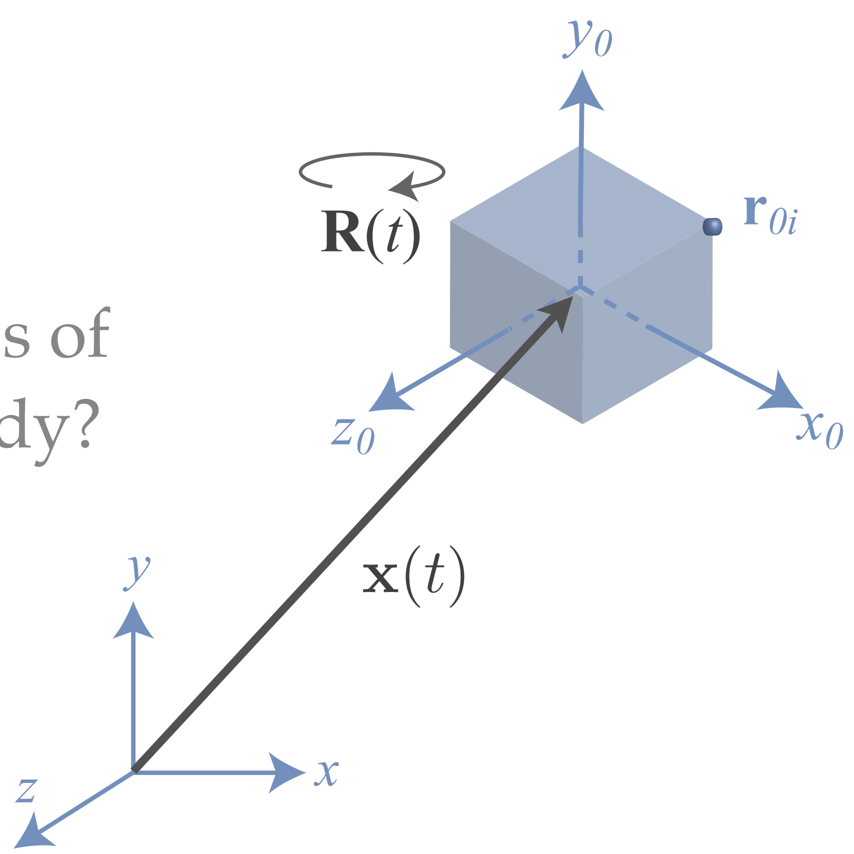
World space



Position and orientation

World space

Use $\mathbf{x}(t)$ and $\mathbf{R}(t)$ to transform the body space into world space



What are the world coordinates of an arbitrary point $\mathbf{r}_{0i}$ on the body?

$$\mathbf{r}_i(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_{0i}$$

Position and orientation

- Assume the rigid body has uniform density, what is the physical meaning of $\mathbf{x}(t)$?
  - center of mass over time
- What is the physical meaning of $\mathbf{R}(t)$?
  - it's a bit tricky

Position and orientation

Consider the x-axis in body space, (1, 0, 0), what is the direction of this vector in world space at time *t*?

$$\mathbf{R}(t)\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}r_{xx}\\r_{xy}\\r_{xz}\end{bmatrix}$$

which is the first column of $\mathbf{R}(t)$

$\mathbf{R}(t)$ represents the directions of x, y, and z axes of the body space in world space at time *t*

Position and orientation

- So $\mathbf{x}(t)$ and $\mathbf{R}(t)$ define the position and the orientation of the body at time t
- Next we need to define how the position and orientation change over time

- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

Linear velocity

Since $\mathbf{x}(t)$ is the position of the center of mass in world space, $\dot{\mathbf{x}}(t)$ is the velocity of the center of mass in world space

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t)$$

Angular velocity

- If we freeze the position of the COM in space
  - then any movement is due to the body spinning about some axis that passes through the COM
  - Otherwise, the COM would itself be moving

Angular velocity

We describe that spin as a vector $\omega(t)$

Direction of $\omega(t)$?

Magnitude of $|\omega(t)|$?

Using this representation, any movement of COM is due to the linear velocity and angular velocity spins the object around COM.

Angular velocity

Linear position and velocity are related by $\mathbf{v}(t) = \frac{d}{dt}\mathbf{x}(t)$

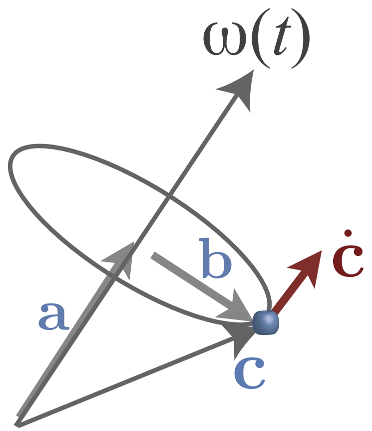
How are angular position (orientation) and velocity related?

Angular velocity

How are $\mathbf{R}(t)$ and $\omega(t)$ related?

Hint:

Consider a vector $\mathbf{c}(t)$ at time t specified in world space, how do we represent $\dot{\mathbf{c}}(t)$ in terms of $\omega(t)$



$$|\dot{\mathbf{c}}(t)| = |\mathbf{b}||\omega(t)| = |\omega(t) \times \mathbf{b}|$$

$$\dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{b} = \omega(t) \times \mathbf{b} + \omega(t) \times \mathbf{a}$$

$$\dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{c}(t)$$

Angular velocity

Given the physical meaning of $\mathbf{R}(t)$, what does each column of $\dot{\mathbf{R}}(t)$ mean?

At time t, the direction of x-axis of the rigid body in world space is the first column of $\mathbf{R}(t)$

$$\begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix}$$

At time t, what is the derivative of the first column of $\mathbf{R}(t)$?

$$\dot{\begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix}} = \omega(t) \times \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix}$$

Angular velocity

$$\dot{\mathbf{R}}(t) = \left[\begin{array}{ccc} \omega(t) \times \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} & \omega(t) \times \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} & \omega(t) \times \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \end{array} \right]$$

This is the relation between angular velocity and the orientation, but it is too cumbersome

We can use a trick to simplify this expression

Angular velocity

Consider two 3 by 1 vectors: **a** and **b**, the cross product of them is

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{bmatrix}$$

Given **a**, let's define **a*** to be a skew symmetric matrix

$$\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

then

$$\mathbf{a}^*\mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \mathbf{a} \times \mathbf{b}$$

Angular velocity

$$\dot{\mathbf{R}}(t) = \left[\omega(t) \times \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \; \omega(t) \times \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \; \omega(t) \times \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right]$$

$$\dot{\mathbf{R}}(t) = \left[\omega(t)^* \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \; \omega(t)^* \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \; \omega(t)^* \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right]$$

$$= \omega(t)^* \mathbf{R}(t)$$

Vector relation: $\dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{c}(t)$

Matrix relation: $\dot{\mathbf{R}} = \omega(t)^* \mathbf{R}(t)$

Perspective of particles

- Imagine a rigid body is composed of a large number of small particles
  - the particles are indexed from 1 to N
  - each particle has a constant location $\mathbf{r}_{0i}$ in body space
  - the location of *i*-th particle in world space at time t is $\mathbf{r}_i(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_{0i}$

Velocity of a particle

$$\dot{\mathbf{r}}(t) = \frac{d}{dt}\mathbf{r}(t) = \omega^*\mathbf{R}(t)\mathbf{r}_{0i} + \mathbf{v}(t)$$

$$= \omega^*(\mathbf{R}(t)\mathbf{r}_{0i} + \mathbf{x}(t) - \mathbf{x}(t)) + \mathbf{v}(t)$$

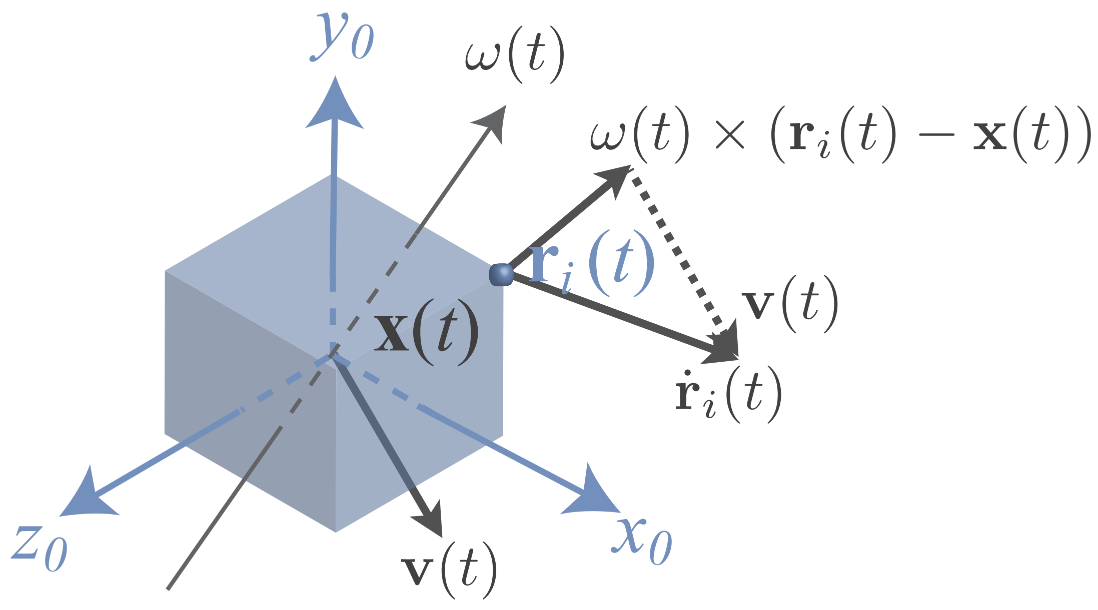
$$= \omega^*(\mathbf{r}_i(t) - \mathbf{x}(t)) + \mathbf{v}(t)$$

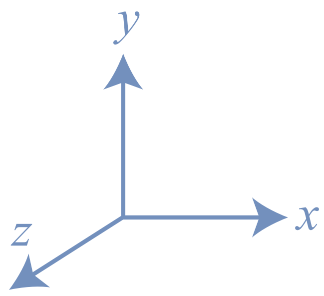
$$\dot{\mathbf{r}}_i(t) = \omega \times (\mathbf{r}_i(t) - \mathbf{x}(t)) + \mathbf{v}(t)$$

angular component   linear component

Velocity of a particle

$$\dot{\mathbf{r}}_i(t) = \omega \times (\mathbf{r}_i(t) - \mathbf{x}(t)) + \mathbf{v}(t)$$





Quiz

- True or False
- If a cube has non-zero angular velocity, a corner point always moves faster than the COM
- If a cube has zero angular velocity, a corner point always moves at the same speed as the COM
- If a cube has non-zero angular velocity and zero linear velocity, the COM may or may not be moving

- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

Mass

The mass of the i-th particle is m_i

Mass $$M = \sum_{i=1}^{N} m_i$$

Center of mass in world space $$\frac{\sum m_i r_i(t)}{M}$$

What about center of mass in body space? (0, 0, 0)

Quiz

Proof that the center of mass at time t in word space is $\mathbf{x}(t)$

$$\frac{\sum m_i \mathbf{r}_i(t)}{M} =$$

$$= \mathbf{x}(t)$$

Inertia tensor

Inertia tensor describes how the mass of a rigid body is distributed relative to the center of mass

$$\mathbf{I} = \sum_i \begin{bmatrix} m_i(r'^2_{iy} + r'^2_{iz}) & -m_i r'_{ix} r'_{iy} & -m_i r'_{ix} r'_{iz} \\ -m_i r'_{iy} r'_{ix} & m_i(r'^2_{ix} + r'^2_{iz}) & -m_i r'_{iy} r'_{iz} \\ -m_i r'_{iz} r'_{ix} & -m_i r'_{iz} r'_{iy} & m_i(r'^2_{ix} + r'^2_{iy}) \end{bmatrix}$$

$$\mathbf{r}'_i = \mathbf{r}_i(t) - \mathbf{x}(t)$$

$\mathbf{I}(t)$ depends on the orientation of a body, but not the translation

For an actual implementation, we replace the finite sum with the integrals over a body's volume in world space

Inertia tensor

- Inertia tensors vary in world space over time
- But are constant in the body space
- Pre-compute the integral part in the body space to save time

Inertia tensor

Pre-compute $\mathbf{I}_{body}$ that does not vary over time

$$\mathbf{I}(t) = \sum m_i \mathbf{r}_i'^T \mathbf{r}_i' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} m_i \mathbf{r}_{ix}'^2 & m_i \mathbf{r}_{ix}' \mathbf{r}_{iy}' & m_i \mathbf{r}_{ix}' \mathbf{r}_{iz}' \\ m_i \mathbf{r}_{iy}' \mathbf{r}_{ix}' & m_i \mathbf{r}_{iy}'^2 & m_i \mathbf{r}_{iy}' \mathbf{r}_{iz}' \\ m_i \mathbf{r}_{iz}' \mathbf{r}_{ix}' & m_i \mathbf{r}_{iz}' \mathbf{r}_{iy}' & m_i \mathbf{r}_{iz}'^2 \end{bmatrix}$$

$$\mathbf{I} = \sum_i \begin{bmatrix} m_i(r_{iy}'^2 + r_{iz}'^2) & -m_i r_{ix}' r_{iy}' & -m_i r_{ix}' r_{iz}' \\ -m_i r_{iy}' r_{ix}' & m_i(r_{ix}'^2 + r_{iz}'^2) & -m_i r_{iy}' r_{iz}' \\ -m_i r_{iz}' r_{ix}' & -m_i r_{iz}' r_{iy}' & m_i(r_{ix}'^2 + r_{iy}'^2) \end{bmatrix}$$

$$\mathbf{r}_i' = \mathbf{r}_i(t) - \mathbf{x}(t)$$

$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{body}\mathbf{R}(t)^T \qquad \mathbf{I}_{body} = \sum_i m_i((\mathbf{r}_{0i}^T \mathbf{r}_{0i})\mathbf{1} - \mathbf{r}_{0i}\mathbf{r}_{0i}^T)$$

Inertia tensor

Pre-compute $\mathbf{I}_{body}$ that does not vary over time

$$\mathbf{I}(t) = \sum m_i \mathbf{r}_i'^T \mathbf{r}_i' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} m_i \mathbf{r}_{ix}'^2 & m_i \mathbf{r}_{ix}' \mathbf{r}_{iy}' & m_i \mathbf{r}_{ix}' \mathbf{r}_{iz}' \\ m_i \mathbf{r}_{iy}' \mathbf{r}_{ix}' & m_i \mathbf{r}_{iy}'^2 & m_i \mathbf{r}_{iy}' \mathbf{r}_{iz}' \\ m_i \mathbf{r}_{iz}' \mathbf{r}_{ix}' & m_i \mathbf{r}_{iz}' \mathbf{r}_{iy}' & m_i \mathbf{r}_{iz}'^2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{I}(t) &= \sum m_i (\mathbf{r}_i'^T \mathbf{r}_i') \mathbf{1} - \mathbf{r}_i' \mathbf{r}_i'^T) \\ &= \sum m_i ((\mathbf{R}(t)\mathbf{r}_{0i})^T (\mathbf{R}(t)\mathbf{r}_{0i}) \mathbf{1} - (\mathbf{R}(t)\mathbf{r}_{0i})(\mathbf{R}(t)\mathbf{r}_{0i})^T) \\ &= \sum m_i (\mathbf{R}(t)(\mathbf{r}_{0i}^T \mathbf{r}_{0i}) \mathbf{R}(t)^T \mathbf{1} - \mathbf{R}(t) \mathbf{r}_{0i} \mathbf{r}_{0i}^T \mathbf{R}(t)^T) \\ &= \mathbf{R}(t) \left(\sum m_i ((\mathbf{r}_{0i}^T \mathbf{r}_{0i}) \mathbf{1} - \mathbf{r}_{0i} \mathbf{r}_{0i}^T) \right) \mathbf{R}(t)^T \end{aligned}$$

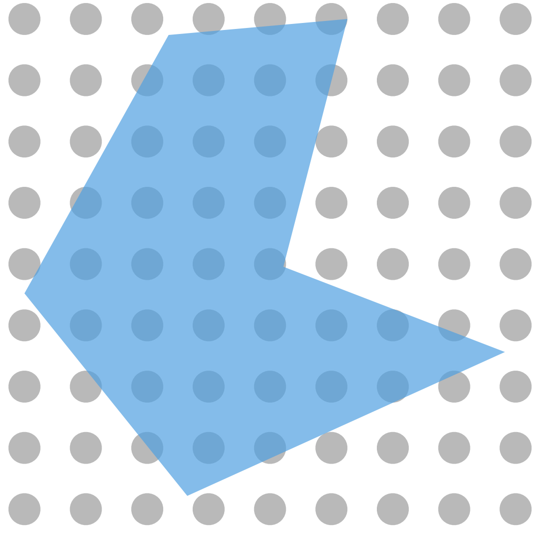
$$\mathbf{I}(t) = \mathbf{R}(t) \mathbf{I}_{body} \mathbf{R}(t)^T \qquad \mathbf{I}_{body} = \sum_i m_i ((\mathbf{r}_{0i}^T \mathbf{r}_{0i}) \mathbf{1} - \mathbf{r}_{0i} \mathbf{r}_{0i}^T)$$

Approximate inertia tensor

- Bounding boxes
  - Pros: simple
  - Cons: inaccurate

Approximate inertia tensor

- Point sampling
  - Pros: simple, fairly accurate
  - Cons: expensive, requires volume test

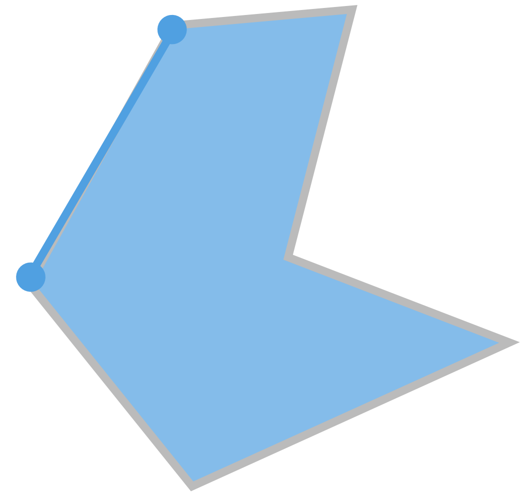


Approximate inertia tensor

- Green's theorem

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int\int_{D} (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

  - Pros: simple, exact
  - Cons: require boundary representation



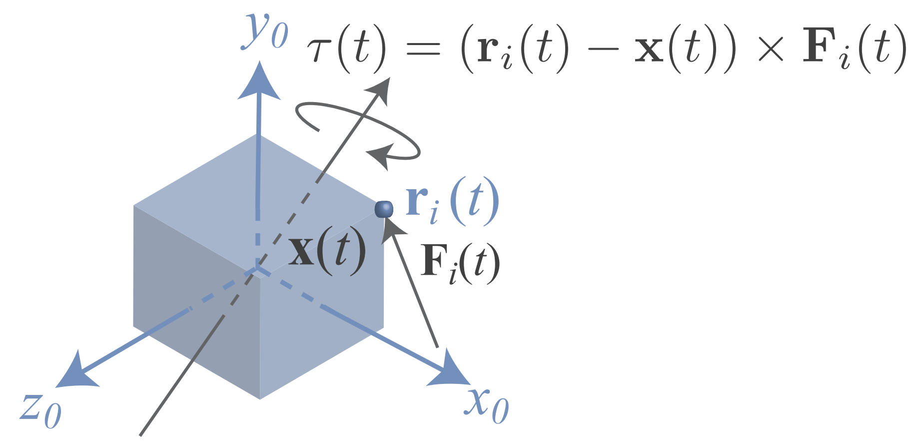
- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

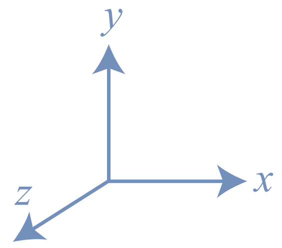
Force and torque

$\mathbf{F}_i(t)$ denotes the total force from external forces acting on the i-th particle at time t

$$\mathbf{F}(t) = \sum_i \mathbf{F}_i(t)$$

$$\tau(t) = \sum_i (\mathbf{r}_i(t) - \mathbf{x}(t)) \times \mathbf{F}_i(t)$$





Force and torque

- $\mathbf{F}(t)$ conveys no information about where the various forces acted on the body
- $\boldsymbol{\tau}(t)$ contains the information about the distribution of the forces over the body
- Which one depends on the location of the particle relative to the center of mass?

Linear momentum

$$\mathbf{P}(t) = \sum_i m_i \dot{r}_i(t)$$

$$= \sum_i m_i \mathbf{v}(t) + \omega(t) \times \sum_i m_i (\mathbf{r}_i(t) - \mathbf{x}(t))$$

$$= M\mathbf{v}(t)$$

Total linear moment of the rigid body is the same as if the body was simply a particle with mass M and velocity $\mathbf{v}(t)$

Angular momentum

Similar to linear momentum, angular momentum is defined as

$$\mathbf{L}(t) = \mathbf{I}(t)\omega(t)$$

Does $\mathbf{L}(t)$ depend on the translational effect $\mathbf{x}(t)$?

Does $\mathbf{L}(t)$ depend on the rotational effect $\mathbf{R}(t)$?

What about $\mathbf{P}(t)$?

Derivative of momentum

Change in linear momentum is equivalent to the total forces acting on the rigid body

$$\dot{\mathbf{P}}(t) = M\dot{\mathbf{v}}(t) = \mathbf{F}(t)$$

The relation between angular momentum and the total torque is analogous to the linear case

$$\dot{\mathbf{L}}(t) = \tau(t)$$

Derivative of momentum

Proof $\dot{\mathbf{L}}(t) = \tau(t) = \sum \mathbf{r}'_i \times \mathbf{F}_i$

$$m_i \ddot{r}_i - \mathbf{F}_i = m_i(\dot{\mathbf{v}} - \dot{\mathbf{r}}'^*_i \omega - \mathbf{r}'^*_i \dot{\omega}) - \mathbf{F}_i = \mathbf{0}$$

$$\sum \mathbf{r}'^*_i m_i(\dot{\mathbf{v}} - \dot{\mathbf{r}}'^*_i \omega - \mathbf{r}'^*_i \dot{\omega}) - \sum \mathbf{r}'^*_i \mathbf{F}_i = \mathbf{0}$$

$$-\left(\sum m_i \mathbf{r}'^*_i \dot{\mathbf{r}}'^*_i\right)\omega - \left(\sum m_i \mathbf{r}'^*_i \mathbf{r}'^*_i\right)\dot{\omega} = \tau$$

$$\sum -m_i \mathbf{r}'^*_i \mathbf{r}'^*_i = \sum m_i((\mathbf{r}'^T_i \mathbf{r}'_i)\mathbf{1} - \mathbf{r}'_i \mathbf{r}'^T_i) = \mathbf{I}(t)$$

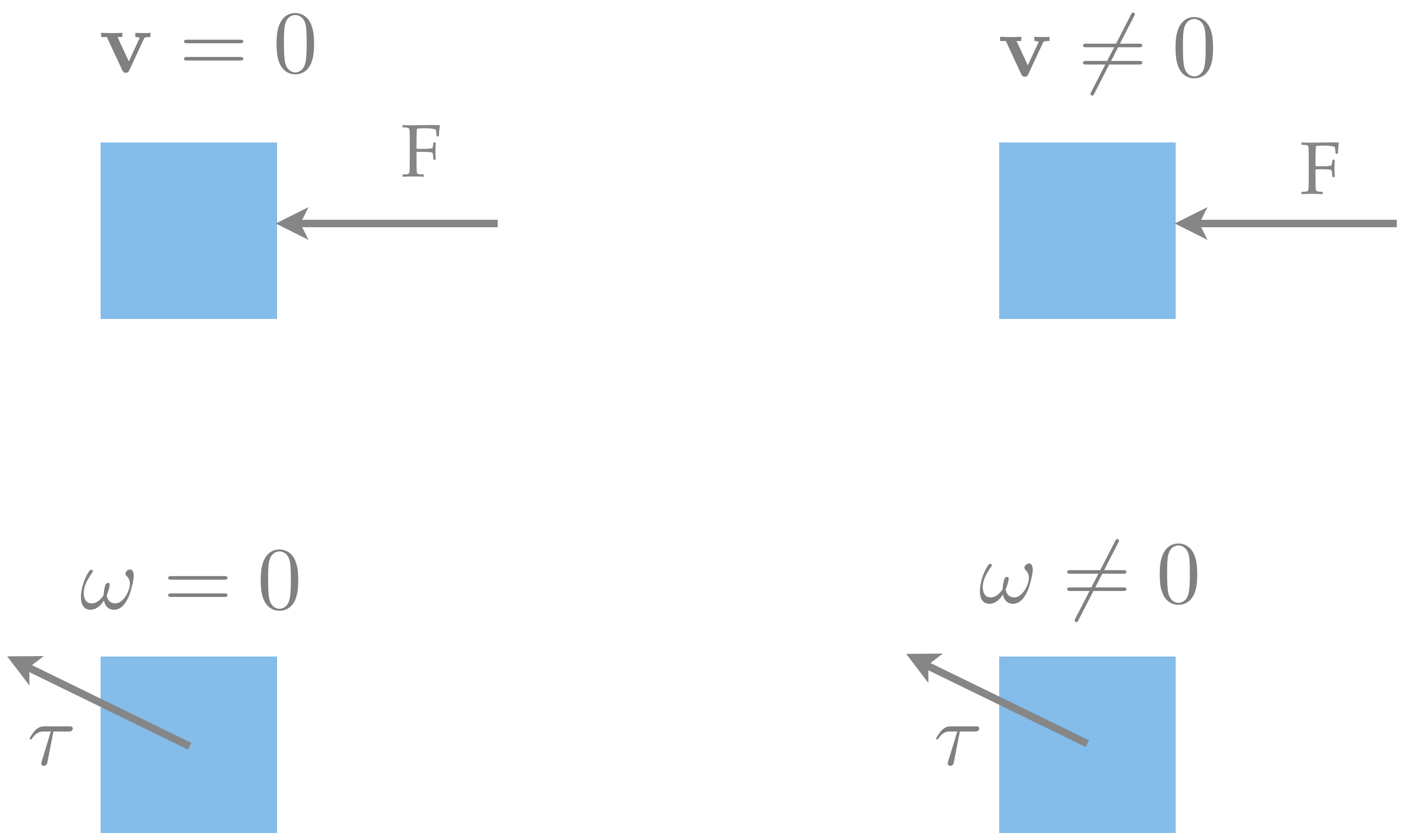
$$-\left(\sum m_i \mathbf{r}'^*_i \dot{\mathbf{r}}'^*_i\right)\omega + \mathbf{I}(t)\dot{\omega} = \tau$$

$$\dot{\mathbf{I}}(t) = \frac{d}{dt}\sum -m_i \mathbf{r}'^*_i \mathbf{r}'^*_i = \sum -m_i \mathbf{r}'^*_i \dot{\mathbf{r}}'^*_i - m_i \dot{\mathbf{r}}'^*_i \mathbf{r}'^*_i$$

$$\dot{\mathbf{I}}(t)\omega + \mathbf{I}(t)\dot{\omega} = \frac{d}{dt}(\mathbf{I}(t)\omega) = \dot{\mathbf{L}}(t) = \tau$$

Quiz

What is the direction of acceleration?



- Position and orientation
- Linear and angular velocity
- Mass and Inertia
- Force and torques
- Simulation

Equation of motion

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{bmatrix} \begin{matrix} \text{position} \\ \text{orientation} \\ \text{linear momentum} \\ \text{angular momentum} \end{matrix}$$

$$\frac{d}{dt}\mathbf{Y}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \omega(t)^*\mathbf{R}(t) \\ \mathbf{F}(t) \\ \tau(t) \end{bmatrix}$$

Constants: M and $\mathbf{I}_{body}$

$$\mathbf{v}(t) = \frac{\mathbf{P}(t)}{M}$$

$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{body}\mathbf{R}(t)^T$$

$$\omega(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t)$$

Issues with 3D orientation

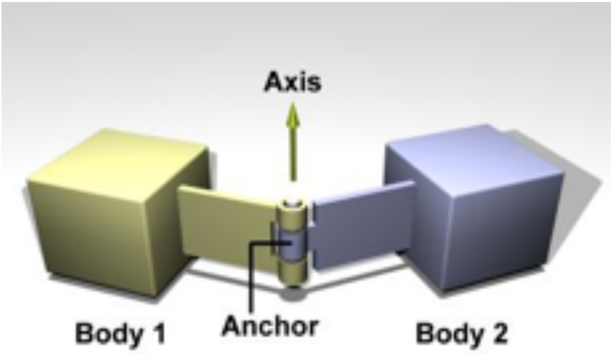
- After simulating for a while, the rotational matrix might no longer be orthonormal due to accumulated numerical errors.
- Rectifying a rotational matrix is not trivial.
- A better representation of 3D orientation is needed.

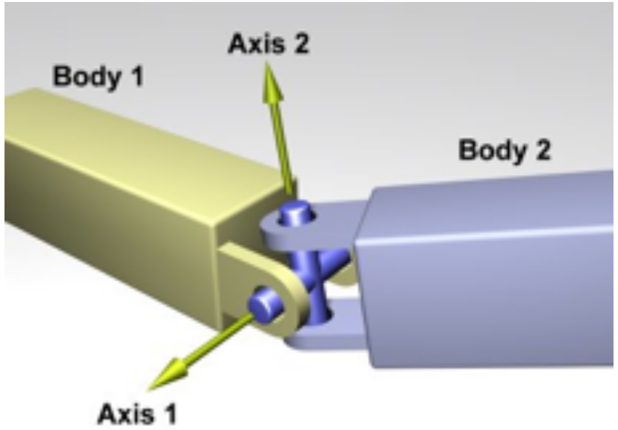
Types of orientation

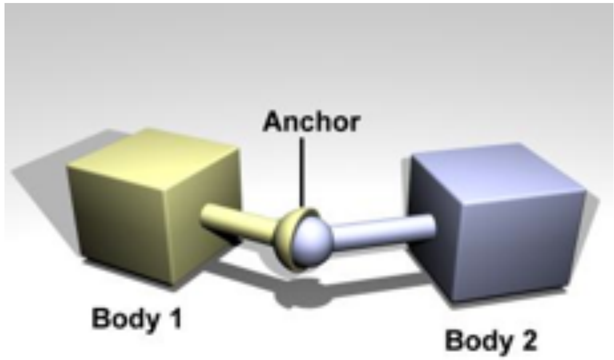
1D orientation

2D orientation

3D orientation







3 Euler-angle representation

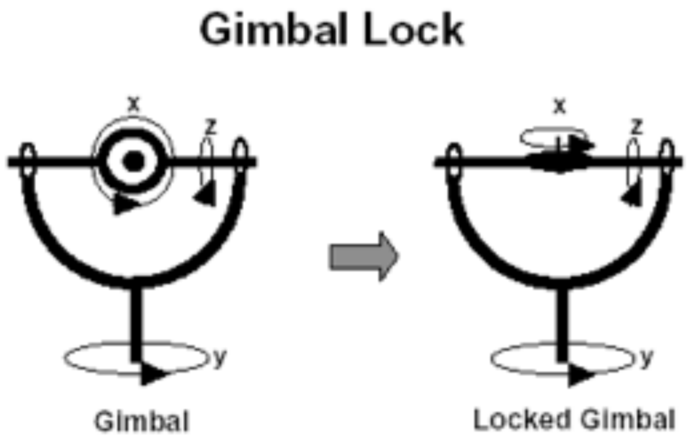
- An Euler angle is a rotation about a single Cartesian axis
- Create multi-DOF rotations by concatenating Euler angles
- Rotate each axis independently in a set order

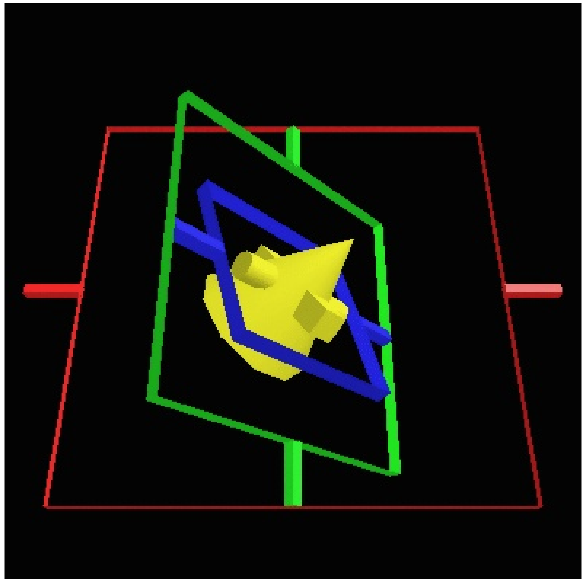
Gimbal Lock

- A Gimbal is a hardware implementation of Euler angles used for mounting gyroscopes or expensive globes
- Gimbal lock is a basic problem with representing 3D rotation using Euler angles or fixed angles

Gimbal lock

- When two rotational axis of an object pointing in the same direction, the rotation ends up losing one degree of freedom

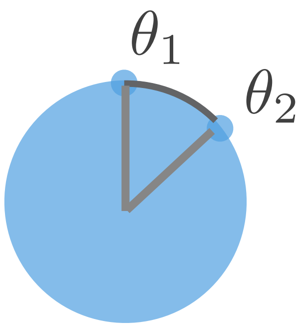




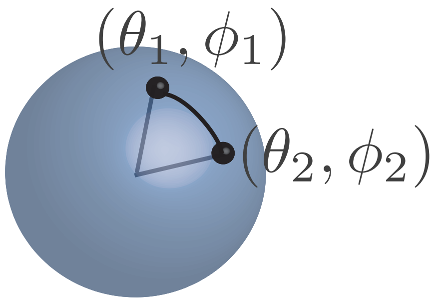
Quaternion

- Quaternion is free from Gimbal lock.
- Quaternion experiences less numerical drift than matrix
- If it does become necessary to account for drift in a quaternion, it is easily correctable by re-normalizing the quaternion to unit length

Quaternion: geometric view



1-angle rotation can be represented by a unit circle



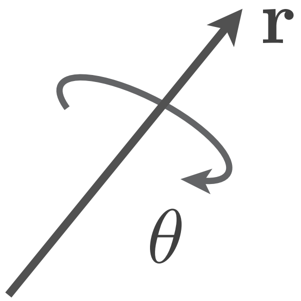
2-angle rotation can be represented by a unit sphere

- What about 3-angle rotation?

Quaternion: algebraic view

- 4 tuple of real numbers: w, x, y, z

$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} \begin{matrix} \text{scalar} \\ \text{vector} \end{matrix}$$



$$\mathbf{q} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathbf{r} \end{bmatrix}$$

Basic quaternion definitions

- Unit quaternion $|\mathbf{q}| = 1$

$$x^2 + y^2 + z^2 + w^2 = 1$$

- Inverse quaternion $\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{|\mathbf{q}|}$

Conjugate $\mathbf{q}^* = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix}^* = \begin{bmatrix} w \\ -\mathbf{v} \end{bmatrix}$

- Identity

$$\mathbf{q}\mathbf{q}^{-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Quaternion multiplication

$$\begin{bmatrix} w_1 \\ \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix}$$

- Commutativity

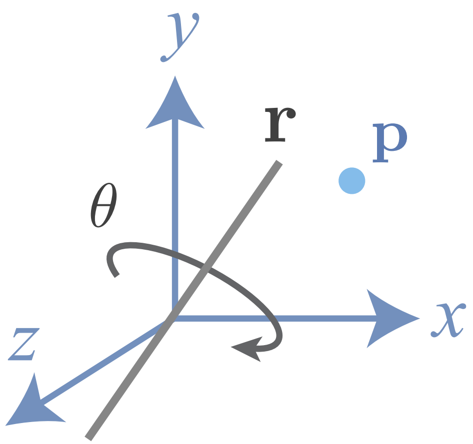
$$\mathbf{q}_1 \mathbf{q}_2 \neq \mathbf{q}_2 \mathbf{q}_1$$

- Associativity

$$\mathbf{q}_1 (\mathbf{q}_2 \mathbf{q}_3) = (\mathbf{q}_1 \mathbf{q}_2) \mathbf{q}_3$$

Quaternion rotation

- Quaternion representation of a 3D point, **p**



$$\mathbf{q}_p = \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathbf{r} \end{bmatrix}$$

- If **q** is a unit quaternion, $\mathbf{q}\mathbf{q}_p\mathbf{q}^{-1}$ results in **p** rotating about **r** by θ

proof: see *Quaternions* by Shoemaker

Quaternion composition

- If $\mathbf{q}_1$ and $\mathbf{q}_2$ are unit quaternions, the combined rotation of first rotating by $\mathbf{q}_1$ and then by $\mathbf{q}_2$ is equivalent to

$$\mathbf{q}_3 = \mathbf{q}_2 \cdot \mathbf{q}_1$$

Quaternion to matrix

$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy + 2wz & 2xz - 2wy & 0 \\ 2xy - 2wz & 1 - 2x^2 - 2z^2 & 2yz + 2wx & 0 \\ 2xz + 2wy & 2yz - 2wx & 1 - 2x^2 - 2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Quaternion derivative

- To represent orientation of rigid body using quaternion, we need to compute time derivative of quaternion

$$\dot{\mathbf{q}}(t) = \frac{1}{2}\boldsymbol{\omega}(t)\mathbf{q}(t) = \frac{1}{2}[0, \boldsymbol{\omega}(t)]\mathbf{q}(t)$$

Modified equations

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{bmatrix} \begin{matrix} \text{position} \\ \text{orientation} \\ \text{linear momentum} \\ \text{angular momentum} \end{matrix}$$

$$\frac{d}{dt}\mathbf{Y}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \omega(t)^*\mathbf{R}(t) \\ \mathbf{F}(t) \\ \tau(t) \end{bmatrix}$$

Constants: M and $\mathbf{I}_{body}$

$$\mathbf{v}(t) = \frac{\mathbf{P}(t)}{M}$$

$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{body}\mathbf{R}(t)^T$$

$$\omega(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t)$$

Modified equations

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \sout{\mathbf{R}(t)}\ \mathbf{q}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{bmatrix} \begin{matrix} \text{position} \\ \text{orientation} \\ \text{linear momentum} \\ \text{angular momentum} \end{matrix}$$

$$\frac{d}{dt}\mathbf{Y}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \sout{\omega(t)^*\mathbf{R}(t)}\ \dot{\mathbf{q}}(t) \\ \mathbf{F}(t) \\ \tau(t) \end{bmatrix}$$

Constants: M and $\mathbf{I}_{body}$

$$\mathbf{v}(t) = \frac{\mathbf{P}(t)}{M}$$

$$\mathbf{R}(t) = \text{quatToMatrix}(\mathbf{q}(t))$$

$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{body}\mathbf{R}(t)^T$$

$$\omega(t) = \mathbf{I}(t)^{-1}\mathbf{L}(t)$$

$$\dot{\mathbf{q}}(t) = \frac{1}{2}\boldsymbol{\omega}(t)\mathbf{q}(t)$$

Numerical errors

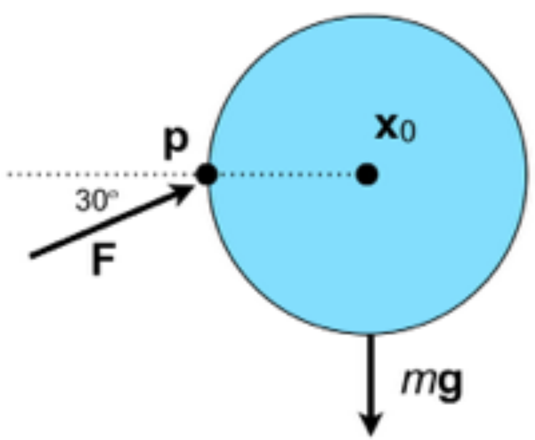
- The numerical errors could build up and $\mathbf{q}(t)$ might no longer be a unit quaternion.
- Normalizing $\mathbf{q}(t)$ is trivial.

Momentum vs. velocity

- Why do we use momentum in the state space instead of velocity?
  - Because the relation of angular momentum and torque is simpler
  - Because the angular momentum is constant when there is no torques acting on the object
  - Use linear momentum $\mathbf{P}(t)$ to be consistent with angular velocity and acceleration

Quiz

Consider a 3D sphere with radius 1m, mass 1kg, and inertia $\mathbf{I}_{body}$. The initial linear and angular velocity are both zero. The initial position and the initial orientation are $\mathbf{x}_0$ and $\mathbf{R}_0$. The forces applied on the sphere include gravity (g) and an initial push $\mathbf{F}$ applied at point $\mathbf{p}$. Note that $\mathbf{F}$ is only applied for one time step at t_0. If we use Explicit Euler method with time step h to integrate , what are the position and the orientation of the sphere at t_2? Use the actual numbers defined as below to compute your solution (except for g and h).



$\mathbf{x}_0 = (0, 0, 0)$

$$\mathbf{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{p} = (-1, 0, 0)$

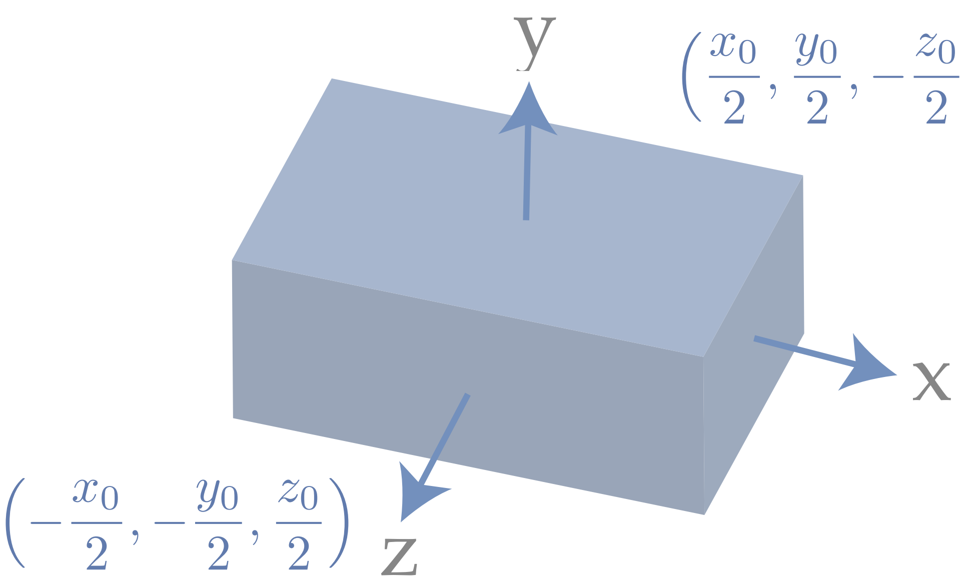
$\mathbf{F} = (4\cos(30^\circ), 4\sin(30^\circ), 0)$

$m = 1$

$$\mathbf{I}_{body} = \begin{pmatrix} 2/5 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/5 \end{pmatrix}$$

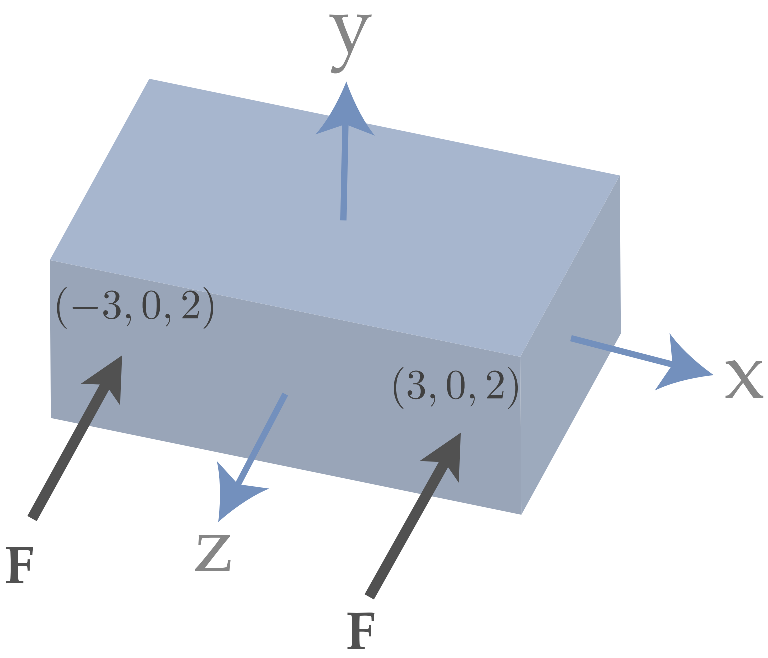
Example:

1. compute the $\mathbf{I}_{body}$ in body space



$$\mathbf{I}_{body} = \frac{M}{12}\begin{pmatrix} y_0^2 + z_0^2 & 0 & 0 \\ 0 & x_0^2 + z_0^2 & 0 \\ 0 & 0 & x_0^2 + y_0^2 \end{pmatrix}$$

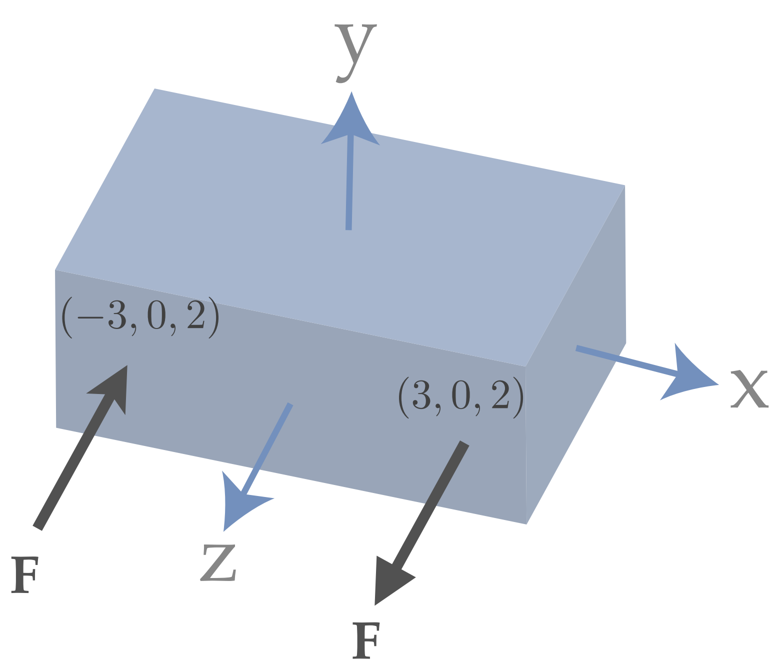
Example:



1. compute the $\mathbf{I}_{body}$ in body space

2. rotation free movement

Example:



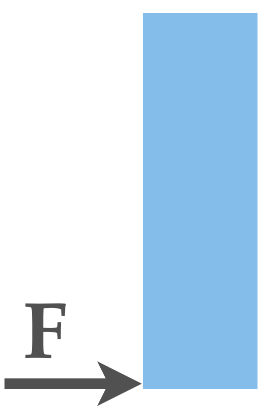
1. compute the $\mathbf{I}_{body}$ in body space
2. rotation free movement
3. translation free movement

Quiz

10 sec later

energy = 0 energy = $\frac{1}{2}M\mathbf{v}^T\mathbf{v}$

Suppose a force F acts on the block at the center of mass for 10 seconds. Since there is no torque acting on the block, the body will only acquire linear velocity **v** after 10 seconds. The kinetic energy will be $\frac{1}{2}M\mathbf{v}^T\mathbf{v}$



energy = 0

Now, consider the same force acting off-center to the body for 10 seconds. Since it is the same force, the velocity of the center of mass after 10 seconds is the same **v**. However, the block will also pick up some angular velocity ω. The kinetic energy will be $\frac{1}{2}M\mathbf{v}^T\mathbf{v} + \frac{1}{2}\omega^T\mathbf{I}\omega$

If identical forces push the block in both cases, how can the energy of the block be different?