Rigid body dynamics
Once we consider an object with spatial extent, particle system simulation is no longer sufficient.
Rigid body simulation

- Unconstrained system
  - no contact
- Constrained system
  - collision and contact
Problems

Performance is important!
Problems

Control is difficult!
Particle simulation

\[ \mathbf{Y}(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \]  
Position in phase space

\[ \dot{\mathbf{Y}}(t) = \begin{bmatrix} v(t) \\ f(t)/m \end{bmatrix} \]  
Velocity in phase space
Rigid body concepts

Translation
- Position
- Linear velocity
- Mass
- Linear momentum
- Force

Rotation
- Orientation
- Angular velocity
- Inertia tensor
- Angular momentum
- Torque
• Position and orientation

• Linear and angular velocity

• Mass and Inertia

• Force and torques

• Simulation
Position and orientation

Translation of the body

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotation of the body

$$\mathbf{R}(t) = \begin{bmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{xz} & r_{yz} & r_{zz} \end{bmatrix}$$

$\mathbf{x}(t)$ and $\mathbf{R}(t)$ are called spatial variables of a rigid body.
Body space

A fixed and unchanged space where the shape of a rigid body is defined

The geometric center of the rigid body lies at the origin of the body space
Position and orientation

Body space

World space

Position and orientation

Body space

World space

x
y
z
x_0
y_0
z_0
r_{0i}

\mathbf{r}_{0i}

\mathbf{x}(t)

R(t)

x_0
y_0
z_0
r_{0i}
Use $x(t)$ and $R(t)$ to transform the body space into world space.

What are the world coordinates of an arbitrary point $r_{0i}$ on the body?

$$r_i(t) = x(t) + R(t)r_{0i}$$
Position and orientation

• Assume the rigid body has uniform density, what is the physical meaning of $x(t)$?
  • center of mass over time
• What is the physical meaning of $R(t)$?
  • it’s a bit tricky
Consider the x-axis in body space, \((1, 0, 0)\), what is the direction of this vector in world space at time \(t\)?

\[
\mathbf{R}(t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix}
\]

which is the first column of \(\mathbf{R}(t)\)

\(\mathbf{R}(t)\) represents the directions of \(x\), \(y\), and \(z\) axes of the body space in world space at time \(t\)
So $x(t)$ and $R(t)$ define the position and the orientation of the body at time $t$.

Next we need to define how the position and orientation change over time.
• Position and orientation
• Linear and angular velocity
• Mass and Inertia
• Force and torques
• Simulation
Linear velocity

Since $x(t)$ is the position of the center of mass in world space, $\dot{x}(t)$ is the velocity of the center of mass in world space.

$$v(t) = \dot{x}(t)$$
Angular velocity

• If we freeze the position of the COM in space
  • then any movement is due to the body spinning about some axis that passes through the COM
  • Otherwise, the COM would itself be moving
Angular velocity

We describe that spin as angular velocity, a vector $\omega(t)$.

Direction of $\omega(t)$?

Magnitude of $|\omega(t)|$?

Using this representation, any movement of COM is due to the linear velocity. Angular velocity only spins the object around COM.
Angular velocity

Linear position and velocity are related by $\mathbf{v}(t) = \frac{d}{dt} \mathbf{x}(t)$

How are angular position (orientation) and velocity related?
Angular velocity

How are $\mathbf{R}(t)$ and $\omega(t)$ related?

**Hint:**

Consider a vector $\mathbf{c}(t)$ at time $t$ specified in world space, how do we represent $\dot{\mathbf{c}}(t)$ in terms of $\omega(t)$?

\[
|\dot{\mathbf{c}}(t)| = |\mathbf{b}| |\omega(t)| = |\omega(t) \times \mathbf{b}|
\]

\[
\dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{b} = \omega(t) \times \mathbf{b} + \omega(t) \times \mathbf{a}
\]

\[
\dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{c}(t)
\]
Angular velocity

Given the physical meaning of $\mathbf{R}(t)$, what does each column of $\dot{\mathbf{R}}(t)$ mean?

At time $t$, the direction of x-axis of the rigid body in world space is the first column of $\mathbf{R}(t)$

$$
\begin{bmatrix}
  r_{xx} \\
  r_{xy} \\
  r_{xz}
\end{bmatrix}
$$

At time $t$, what is the derivative of the first column of $\mathbf{R}(t)$?

$$
\begin{bmatrix}
  \dot{r}_{xx} \\
  \dot{r}_{xy} \\
  \dot{r}_{xz}
\end{bmatrix} = \mathbf{\omega}(t) \times 
\begin{bmatrix}
  r_{xx} \\
  r_{xy} \\
  r_{xz}
\end{bmatrix}
$$
Angular velocity

\[ \dot{R}(t) = \begin{bmatrix} \omega(t) \times \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \\ \omega(t) \times \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \\ \omega(t) \times \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \end{bmatrix} \]

This is the relation between angular velocity and the orientation, but it is too cumbersome.

We can use a trick to simplify this expression.
Angular velocity

Consider two 3 by 1 vectors: \( \mathbf{a} \) and \( \mathbf{b} \), the cross product of them is

\[
\mathbf{a} \times \mathbf{b} = \begin{bmatrix}
  a_y b_z - b_y a_z \\
  -a_x b_z + b_x a_z \\
  a_x b_y - b_x a_y
\end{bmatrix}
\]

Given \( \mathbf{a} \), let's define \( \mathbf{a}^* \) to be a skew symmetric matrix

\[
\begin{bmatrix}
  0 & -a_z & a_y \\
  a_z & 0 & -a_x \\
  -a_y & a_x & 0
\end{bmatrix}
\]

then

\[
\mathbf{a}^* \mathbf{b} = \begin{bmatrix}
  0 & -a_z & a_y \\
  a_z & 0 & -a_x \\
  -a_y & a_x & 0
\end{bmatrix} \begin{bmatrix}
  b_x \\
  b_y \\
  b_z
\end{bmatrix} = \mathbf{a} \times \mathbf{b}
\]
Angular velocity

\[
\dot{\mathbf{R}}(t) = \begin{bmatrix}
\omega(t) \times \begin{pmatrix}
    r_{xx} \\
    r_{xy} \\
    r_{xz}
\end{pmatrix}
\end{bmatrix}
\begin{bmatrix}
\omega(t) \times \begin{pmatrix}
    r_{yx} \\
    r_{yy} \\
    r_{yz}
\end{pmatrix}
\end{bmatrix}
\begin{bmatrix}
\omega(t) \times \begin{pmatrix}
    r_{zx} \\
    r_{zy} \\
    r_{zz}
\end{pmatrix}
\end{bmatrix}
\]

= \omega(t) \times \mathbf{R}(t)

Vector relation: \( \dot{\mathbf{c}}(t) = \omega(t) \times \mathbf{c}(t) \)

Matrix relation: \( \dot{\mathbf{R}} = \omega(t) \times \mathbf{R}(t) \)
Imagine a rigid body is composed of a large number of small particles.

- The particles are indexed from 1 to N.
- Each particle has a constant location $r_{0i}$ in body space.
- The location of $i$-th particle in world space at time $t$ is $r_i(t) = x(t) + R(t)r_{0i}$.
Velocity of a particle

\[ \dot{\mathbf{r}}(t) = \frac{d}{dt} \mathbf{r}(t) = \omega^* \mathbf{R}(t) \mathbf{r}_{0i} + \mathbf{v}(t) \]

\[ = \omega^* (\mathbf{R}(t) \mathbf{r}_{0i} + \mathbf{x}(t) - \mathbf{x}(t)) + \mathbf{v}(t) \]

\[ = \omega^* (\mathbf{r}_i(t) - \mathbf{x}(t)) + \mathbf{v}(t) \]

\[ \dot{\mathbf{r}}_i(t) = \omega \times (\mathbf{r}_i(t) - \mathbf{x}(t)) + \mathbf{v}(t) \]

angular component   linear component
\[ \dot{\mathbf{r}}_i(t) = \omega \times (\mathbf{r}_i(t) - \mathbf{x}(t)) + \mathbf{v}(t) \]
Quiz

• True or False

• If a cube has non-zero angular velocity, a corner point always moves faster than the COM

• If a cube has zero angular velocity, a corner point always moves at the same speed as the COM

• If a cube has non-zero angular velocity and zero linear velocity, the COM may or may not be moving
• Position and orientation
• Linear and angular velocity
• Mass and Inertia
• Force and torques
• Simulation
Mass

The mass of the $i$-th particle is $m_i$

Mass

$$M = \sum_{i=1}^{N} m_i$$

Center of mass in world space

$$\frac{\sum m_i r_i(t)}{M}$$

What about center of mass in body space? (0, 0, 0)
Proof that the center of mass at time $t$ in word space is $x(t)$

$$\frac{\sum m_i r_i(t)}{M} = x(t)$$
Inertia tensor describes how the mass of a rigid body is distributed relative to the center of mass

\[
\mathbf{I} = \sum_i \begin{bmatrix}
    m_i (r_{iy}'^2 + r_{iz}'^2) & -m_i r_{ix}' r_{iy}' & -m_i r_{ix}' r_{iz}' \\
    -m_i r_{iy}' r_{ix}' & m_i (r_{ix}'^2 + r_{iz}'^2) & -m_i r_{iy}' r_{iz}' \\
    -m_i r_{iz}' r_{ix}' & -m_i r_{iz}' r_{iy}' & m_i (r_{ix}'^2 + r_{iy}'^2)
\end{bmatrix}
\]

\[
r_i' = r_i(t) - x(t)
\]

\(\mathbf{I}(t)\) depends on the orientation of a body, but not the translation.

For an actual implementation, we replace the finite sum with the integrals over a body’s volume in world space.
Inertia tensor

- Inertia tensors vary in world space over time
- But are constant in the body space
- Pre-compute the integral part in the body space to save time
Pre-compute $I_{body}$ that does not vary over time

\[
I(t) = \sum m_i r_i'T_i r_i' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} m_i r_{ix}'^2 & m_i r_{ix}' r_{iy}' & m_i r_{ix}' r_{iz}' \\ m_i r_{iy}' r_{ix}' & m_i r_{iy}'^2 & m_i r_{iy}' r_{iz}' \\ m_i r_{iz}' r_{ix}' & m_i r_{iz}' r_{iy}' & m_i r_{iz}'^2 \end{bmatrix}
\]

\[
I = \sum \begin{bmatrix} m_i (r_{iy}'^2 + r_{iz}'^2) & -m_i r_{ix}' r_{iy}' & -m_i r_{ix}' r_{iz}' \\ -m_i r_{iy}' r_{ix}' & m_i (r_{ix}'^2 + r_{iz}'^2) & -m_i r_{iy}' r_{iz}' \\ -m_i r_{iz}' r_{ix}' & -m_i r_{iz}' r_{iy}' & m_i (r_{ix}'^2 + r_{iy}'^2) \end{bmatrix}
\]

\[
r_i' = r_i(t) - x(t)
\]

\[
I(t) = R(t) I_{body} R(t)^T \quad I_{body} = \sum_i m_i ((r_0'r_{0i})^T 1 - r_{0i} r_0'^T)
\]
Inertia tensor

Pre-compute $I_{body}$ that does not vary over time

$$I(t) = \sum m_i r_i'^T r_i' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} m_i r_i'^2 & m_i r_i' r_i' & m_i r_i' r_i' \\ m_i r_i' r_i' & m_i r_i'^2 & m_i r_i' r_i' \\ m_i r_i' r_i' & m_i r_i' r_i' & m_i r_i'^2 \end{bmatrix}$$

$$I(t) = \sum m_i (r_i'^T r_i') \mathbf{1} - r_i'^T r_i'^T$$

$$I(t) = \sum m_i ((R(t)r_{0i})^T (R(t)r_{0i}) \mathbf{1} - (R(t)r_{0i})(R(t)r_{0i})^T)$$

$$I(t) = \sum m_i (R(t)(r_{0i}^T r_{0i})R(t)^T 1 - R(t)r_{0i}r_{0i}^T R(t)^T)$$

$$I(t) = R(t) \left( \sum m_i ((r_{0i}^T r_{0i}) \mathbf{1} - r_{0i} r_{0i}^T) \right) R(t)^T$$

$$I(t) = R(t)I_{body} R(t)^T \quad I_{body} = \sum_i m_i ((r_{0i}^T r_{0i}) \mathbf{1} - r_{0i} r_{0i}^T)$$
Approximate inertia tensor

- Bounding boxes
- Pros: simple
- Cons: inaccurate
Approximate inertia tensor

- Point sampling
  - Pros: simple, fairly accurate
  - Cons: expensive, requires volume test
Approximate inertia tensor

- Green’s theorem

\[ \int_{\partial D} \mathbf{F} \cdot ds = \int \int_D (\nabla \times \mathbf{F}) \cdot d\mathbf{a} \]

- Pros: simple, exact

- Cons: require boundary representation
• Position and orientation
• Linear and angular velocity
• Mass and Inertia
• Force and torques
• Simulation
Force and torque

$F_i(t)$ denotes the total force from external forces acting on the $i$-th particle at time $t$

\[
F(t) = \sum_i F_i(t)
\]

\[
\tau(t) = \sum_i (r_i(t) - x(t)) \times F_i(t)
\]
Force and torque

- $F(t)$ conveys no information about where the various forces acted on the body
- $\tau(t)$ contains the information about the distribution of the forces over the body
- Which one depends on the location of the particle relative to the center of mass?
Linear momentum

\[ P(t) = \sum_i m_i \dot{r}_i(t) \]
\[ = \sum_i m_i v(t) + \omega(t) \times \sum_i m_i (\mathbf{r}_i(t) - \mathbf{x}(t)) \]
\[ = M \mathbf{v}(t) \]

Total linear moment of the rigid body is the same as if the body was simply a particle with mass \( M \) and velocity \( \mathbf{v}(t) \)
Similar to linear momentum, angular momentum is defined as

\[ \mathbf{L}(t) = \mathbf{I}(t)\mathbf{\omega}(t) \]

Does \( \mathbf{L}(t) \) depend on the translational effect \( \mathbf{x}(t) \)?
Does \( \mathbf{L}(t) \) depend on the rotational effect \( \mathbf{R}(t) \)?
What about \( \mathbf{P}(t) \)?
Derivative of momentum

Change in linear momentum is equivalent to the total forces acting on the rigid body

\[ \dot{\mathbf{P}}(t) = M \dot{\mathbf{v}}(t) = \mathbf{F}(t) \]

The relation between angular momentum and the total torque is analogous to the linear case

\[ \dot{\mathbf{L}}(t) = \tau(t) \]
Proof $\dot{L}(t) = \tau(t) = \sum \mathbf{r}'_i \times \mathbf{F}_i$

$m_i \ddot{r}_i - \mathbf{F}_i = m_i (\dot{v} - \dot{r}'_i \omega - r' \dot{r}' \omega) - \mathbf{F}_i = 0$

$\sum r'_i m_i (\dot{v} - \dot{r}'_i \omega - r' \dot{r}' \omega) - \sum r'_i \mathbf{F}_i = 0$

$- (\sum m_i r'_i \dot{r}'_i) \omega - (\sum m_i r'_i \dot{r}'_i) \dot{\omega} = \tau$

$\sum -m_i r'_i \dot{r}'_i = \sum m_i ((r'_i T r'_i) \mathbf{1} - r'_i r'_i T) = I(t)$

$- (\sum m_i r'_i \dot{r}'_i) \omega + I(t) \dot{\omega} = \tau$

$\dot{I}(t) = \frac{d}{dt} \sum -m_i r'_i \dot{r}'_i = \sum -m_i r'_i \dot{r}'_i - m_i \dot{r}'_i \dot{r}'_i$

$\dot{I}(t) \omega + I(t) \dot{\omega} = \frac{d}{dt} (I(t) \omega) = \dot{L}(t) = \tau$
What is the direction of acceleration?

\[ \mathbf{v} = 0 \]

\[ \mathbf{v} \neq 0 \]

\[ \omega = 0 \]

\[ \omega \neq 0 \]
• Position and orientation
• Linear and angular velocity
• Mass and Inertia
• Force and torques
• Simulation
Equation of motion

\[
Y(t) = \begin{bmatrix}
  x(t) \\
  R(t) \\
  P(t) \\
  L(t)
\end{bmatrix}
\]

- position
- orientation
- linear momentum
- angular momentum

\[
\frac{d}{dt} Y(t) = \begin{bmatrix}
v(t) \\
\omega(t) \times R(t) \\
F(t) \\
\tau(t)
\end{bmatrix}
\]

Constants: \(M\) and \(I_{\text{body}}\)

\[
v(t) = \frac{P(t)}{M}
\]

\[
I(t) = R(t)I_{\text{body}}R(t)^T
\]

\[
\omega(t) = I(t)^{-1}L(t)
\]
Issues with 3D orientation

- The rotational matrix might no longer be orthonormal due to accumulated numerical errors.
- Rectifying a rotational matrix is not trivial.
Quaterrion representation

- Quaternion experiences less numerical drift than matrix.
- If it does become necessary to account for drift in a quaternion, it is easily correctable by renormalizing the quaternion to unit length.
Quateneion multiplication

\[
\begin{bmatrix}
    w_1 \\
    v_1
\end{bmatrix}
\begin{bmatrix}
    w_2 \\
    v_2
\end{bmatrix}
= 
\begin{bmatrix}
    w_1 w_2 - v_1 \cdot v_2 \\
    w_1 v_2 + w_2 v_1 + v_1 \times v_2
\end{bmatrix}
\]

- Commutativity

\[q_1 q_2 \neq q_2 q_1\]

- Associativity

\[q_1 (q_2 q_3) = (q_1 q_2) q_3\]
Quatenvion derivative

- To represent orientation of rigid body using quaternion, we need to compute time derivative of quaternion

\[ \dot{q}(t) = \frac{1}{2} \omega(t) q(t) = \frac{1}{2} [0, \omega(t)] q(t) \]
Quaternion to matrix

\[ q = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \]

\[ R(q) = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy + 2wz & 2xz - 2wy & 0 \\ 2xy - 2wz & 1 - 2x^2 - 2z^2 & 2yz + 2wx & 0 \\ 2xz + 2wy & 2yz - 2wx & 1 - 2x^2 - 2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
**Modified equations**

\[ \mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{P}(t) \\ \mathbf{L}(t) \end{bmatrix} \]

- position
- orientation
- linear momentum
- angular momentum

\[ \frac{d}{dt} \mathbf{Y}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \omega(t) \mathbf{R}(t) \\ \mathbf{F}(t) \\ \mathbf{\tau}(t) \end{bmatrix} \]

**Constants:** \( M \) and \( I_{body} \)

\[ \mathbf{v}(t) = \frac{\mathbf{P}(t)}{M} \]

\[ \mathbf{I}(t) = \mathbf{R}(t) I_{body} \mathbf{R}(t)^T \]

\[ \omega(t) = \mathbf{I}(t)^{-1} \mathbf{L}(t) \]
Modified equations

\[
\mathbf{Y}(t) = \begin{bmatrix}
\mathbf{x}(t) \\
\mathbf{R}(t) \\
\mathbf{p}(t) \\
\mathbf{l}(t)
\end{bmatrix}
\]

- position
- orientation
- linear momentum
- angular momentum

\[
\frac{d}{dt} \mathbf{Y}(t) = \begin{bmatrix}
\mathbf{v}(t) \\
\mathbf{\omega}(t) \times \mathbf{R}(t) \\
\mathbf{f}(t) \\
\mathbf{\tau}(t)
\end{bmatrix}
\]

Constants: \( M \) and \( I_{\text{body}} \)

\[
\mathbf{v}(t) = \frac{\mathbf{p}(t)}{M}
\]

\[
\mathbf{R}(t) = \text{quatToMatrix}(\mathbf{q}(t))
\]

\[
\mathbf{I}(t) = \mathbf{R}(t) I_{\text{body}} \mathbf{R}(t)^T
\]

\[
\mathbf{\omega}(t) = \mathbf{I}(t)^{-1} \mathbf{l}(t)
\]

\[
\dot{\mathbf{q}}(t) = \frac{1}{2} \mathbf{\omega}(t) \mathbf{q}(t)
\]
Quaternion

\[ \mathbf{q}(t) = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \]

\[ \dot{\mathbf{q}}(t) = \frac{1}{2} \begin{bmatrix} 0 \\ \omega(t) \end{bmatrix} \mathbf{q}(t) \]

quaternion multiplication
Momentum vs. velocity

- Why do we use momentum in the state space instead of velocity?
  - Because the relation of angular momentum and torque is simpler
  - Because the angular momentum is constant when there is no torques acting on the object
  - Use linear momentum $P(t)$ to be consistent with angular velocity and acceleration
Consider a 3D sphere with radius 1m, mass 1kg, and inertia $I_{\text{body}}$. The initial linear and angular velocity are both zero. The initial position and the initial orientation are $x_0$ and $R_0$. The forces applied on the sphere include gravity ($g$) and an initial push $F$ applied at point $p$. Note that $F$ is only applied for one time step at $t_0$. If we use Explicit Euler method with time step $h$ to integrate, what are the position and the orientation of the sphere at $t_2$? Use the actual numbers defined as below to compute your solution (except for $g$ and $h$).
Example:

1. compute the $I_{\text{body}}$ in body space

$$I_{\text{body}} = \frac{M}{12} \begin{pmatrix} y_0^2 + z_0^2 & 0 & 0 \\ 0 & x_0^2 + z_0^2 & 0 \\ 0 & 0 & x_0^2 + y_0^2 \end{pmatrix}$$
Example:

1. compute the $I_{body}$ in body space

2. rotation free movement
Example:

1. compute the $I_{body}$ in body space
2. rotation free movement
3. translation free movement
Force vs. torque puzzle

Suppose a force $F$ acts on the block at the center of mass for 10 seconds. Since there is no torque acting on the block, the body will only acquire linear velocity $v$ after 10 seconds. The kinetic energy will be $\frac{1}{2} Mv^T v$.

Now, consider the same force acting off-center to the body for 10 seconds. Since it is the same force, the velocity of the center of mass after 10 seconds is the same $v$. However, the block will also pick up some angular velocity $\omega$. The kinetic energy will be $\frac{1}{2} Mv^T v + \frac{1}{2} \omega^T I \omega$.

If identical forces push the block in both cases, how can the energy of the block be different?