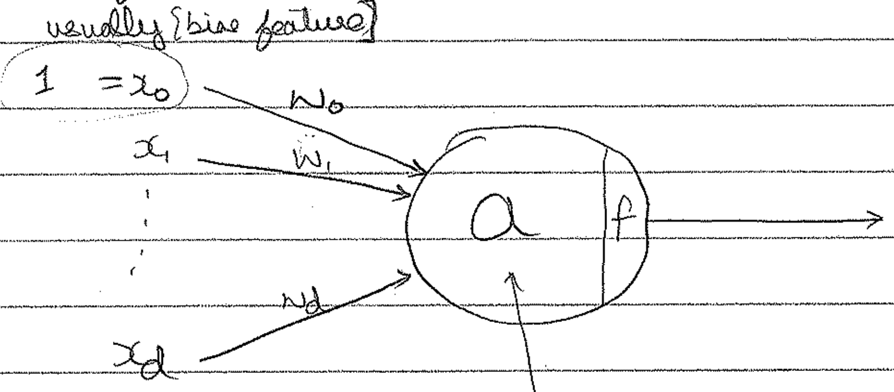
2 Artificial "Neuron"



$\hat{y} = f(a) = f(\vec{w}^T\vec{x})$

Activation / Response Function

Inputs — Strengths of connections — Activation

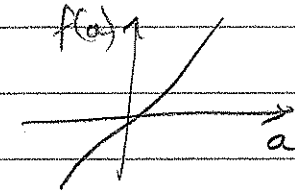
$$a = \sum_{j=0}^{d} w_j x_j$$

$$= \vec{w}^T\vec{x}$$

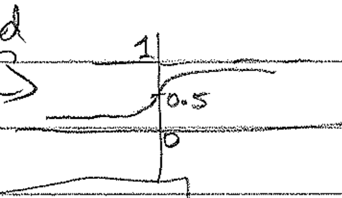
Many different activation functions

→ Linear: $f(a) = a$

$\Rightarrow \hat{y} = \vec{w}^T\vec{x}$ [Linear Regression]



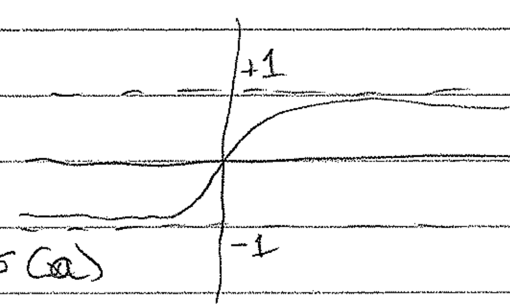
→ Logistic $f(a) = \frac{1}{1+e^{-a}} = \underbrace{\sigma(a)}_{\text{sigmoid}}$



$\hat{y} = \frac{1}{1+e^{\vec{w}^T\vec{x}}}$ | Logistic Regression $P(Y=1 \mid \vec{x}, \vec{w})$

→ Tanh

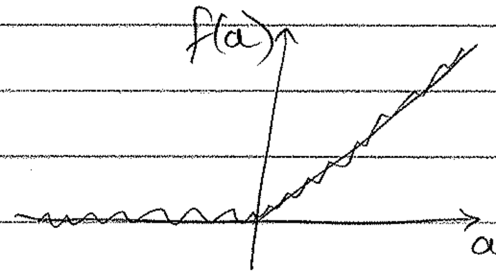
$$f(a) = \frac{e^{a} - e^{-a}}{e^{a} + e^{-a}}$$



For hidden units in NN
we always prefer tanh over $\sigma(a)$
why?

→ ReLU [Rectified Linear Unit]

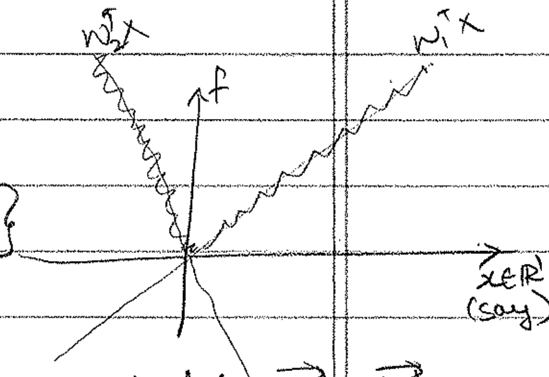
$$f(a) = \max\{0, a\}$$



In hidden layer of deep NN, this is always preferred over $\sigma(a)$ or $\tanh(a)$. Why??

→ Maxout

$$f(a) = \max\{\overbrace{\vec{w}_1^T x}^{a_1}, \overbrace{\vec{w}_2^T x}^{a_2}\}$$



Each neuron has (say) 2 weights $\vec{w}_1, \vec{w}_2$

Take max activation.

ReLU is a special case of this. How?

3 Loss Functions

→ functions of both parameters $\vec{w}$ & training data

1) Log-Loss / Cross-Entropy / Maximum-Likelihood / KL-Divergence

$$L(\vec{w}; D) = \sum_{i=1}^{N} L_i(\vec{w}) \quad \Big\} \text{ Decomposable Loss}$$

$$\text{where} \quad L_i(\vec{w}) = -\log \underbrace{p(y_i^{gt} \mid \vec{x}_i, \vec{w})}_{\substack{\text{How much prob. does your}\\ \text{model assign to GT label?}}}$$

$\equiv$ negative log-likelihood for this sample

→ Why is this called Cross-Entropy? And where is the KL divergence coming in?

Consider Multiclass-classification w/ 1-HOT encoding

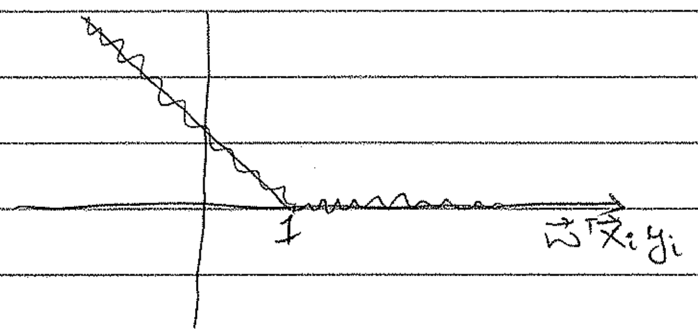
$$\underbrace{\overset{y^{gt}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{1\times K}}}_{\substack{p^{gt}(y) \\ [\text{delta distribution}]}} \qquad \underbrace{\overset{\text{model prediction}}{\begin{bmatrix} p(y=1 \mid \vec{x}, \vec{w}) \\ \vdots \\ p(y=K \mid \vec{x}, \vec{w}) \end{bmatrix}}}_{\substack{\hat{P}(y) \\ [\text{Model distribution}]}}$$

$$KL(p^{gt} \| \hat{p}) = -\sum_{y=1}^{K} p^{gt}(y) \log \hat{p}(y)$$

$$\equiv -\log p(y = y_i^{gt} \mid \vec{x}_i, \vec{w})$$

(2) Hinge-Loss [for binary-classification]

$L_i(\vec{w}) = \max\{0, 1 - \vec{w}^T \vec{x}_i y_i\}$ where $y_i \in \{+1, -1\}$



4 Detour: Matrix/Vector differentiation

	S	V	M
S	$\frac{\partial y}{\partial x}$	$\frac{\partial y}{\partial \vec{x}}$	$\frac{\partial y}{\partial X}$
V	$\frac{\partial \vec{y}}{\partial x}$	$\frac{\partial \vec{y}}{\partial \vec{x}}$	
M	$\frac{\partial Y}{\partial x}$	Tensors	

$x, y \in \mathbb{R}^1$

$\vec{x} \in \mathbb{R}^d$

$\vec{y} \in \mathbb{R}^K$

Convention: $\frac{\partial \vec{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \vdots \\ \frac{\partial y_K}{\partial x} \end{bmatrix}$ numerator = dim 1 = col-vector

[Gradient] $\frac{\partial y}{\partial \vec{x}} = \left[\frac{\partial y}{\partial x_1} \cdots \frac{\partial y}{\partial x_d} \right]$ denominator = dim 2 = row-vector

[Jacobian Matrix] $\frac{\partial \vec{y}}{\partial \vec{x}} = \begin{array}{cc} & j \\ i & \left[\cdots \frac{\partial y_i}{\partial x_j} \right]_{K \times d} \end{array}$

Easy to prove: $\rightarrow \frac{\partial (\vec{w}^T \vec{x})}{\partial \vec{w}} = \left[\frac{\partial (\vec{w}^T \vec{x})}{\partial w_1} \cdots \frac{\partial (\vec{w}^T \vec{x})}{\partial w_d} \right]$

$= X^T$

$\rightarrow \frac{\partial (\vec{w}^T A \vec{w})}{\partial \vec{w}} = 2W^T A$

$\rightarrow \quad \vec{y} = A\vec{x} \qquad \frac{\partial \vec{y}}{\partial \vec{x}} = A$

⑤ Chain Rule

some books/people use opposite order

→ Function Composition: $L(x) = (f \circ g)(x)$
$$= f(g(x))$$

Chain Rule:

→ [Most General Notation]
$$\underbrace{D_x}_{\text{total derivative}} (f \circ g) = D_{g(x)} f \circ D_x g$$

→ [More concrete notation for scalars]
$$L'(x) = f'(g(x))\, g'(x)$$

→ [With intermediate variables]
$$y = g(x)$$
$$z = f(y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

Example: $L_i(w) = -\log\left(\frac{1}{1+e^{-\vec{w}^T x_i}}\right)$ [For $y_i = +1$]

$p \leftarrow \quad a \leftarrow$

$$= \Big(\underbrace{-\log(\cdot)}_{\frac{\partial L}{\partial p}} \circ \underbrace{\frac{1}{1+e^{-(\cdot)}}}_{\frac{\partial p}{\partial a}} \circ \underbrace{x^T(\cdot)}_{\frac{\partial a}{\partial \vec{w}}}\Big)(w)$$

$$\frac{\partial L_i}{\partial \vec{w}} = \left[\frac{1}{p}\right] \cdot \left[\underbrace{\frac{-1}{(1+e^{-a})^2} \cdot -e^{-a}}_{p \cdot (1-p)}\right] \cdot x^T = (1-p)x^T$$

→ Multivariate Chain Rule

$$g: \mathbb{R}^d \to \mathbb{R}^m$$
$$f: \mathbb{R}^m \to \mathbb{R}^k$$

$$L(\vec{x}) = (f \circ g)(\vec{x})$$

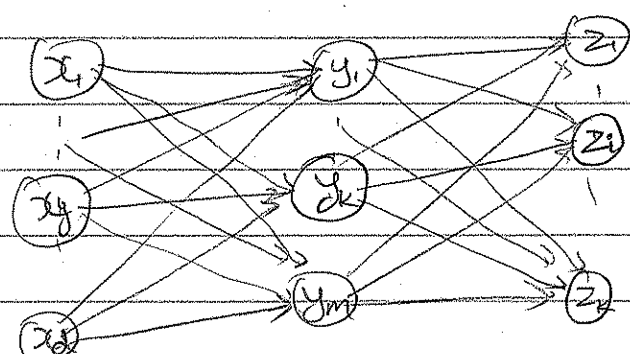
$$\vec{y} = g(\vec{x}) \qquad \vec{z} = f(\vec{y})$$

→ Chain Rule: $D_{\vec{x}}(f \circ g) = D_{g(\vec{x})} f \circ D_{\vec{x}}(g)$

[Abstract form holds]

But what does this mean??

Visualize:



"outcome"

$$\frac{\partial z_i}{\partial x_j} = \sum_k \frac{\partial z_i}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_j}$$

"knob"

how my "knob" affects intermediate variable

all intermediate variables

how those intermediate vars affect outcome!

Formally,

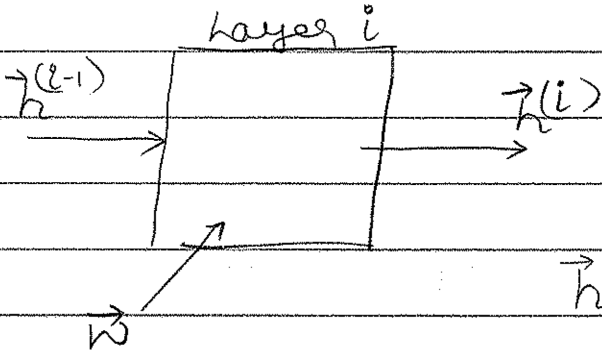
Jacobian relationship

$$J_{f\circ g} = (J_f \circ g)\, J_g$$

$$
\begin{array}{cc} & j \\ i & \left[\begin{array}{c} \cdots \frac{\partial z_i}{\partial x_j} \end{array} \right]_{k\times d} \end{array}
=
\begin{array}{cc} & k \\ i & \left[\begin{array}{c} \cdots \frac{\partial z_i}{\partial y_k} \end{array} \right]_{k\times m} \end{array}
\begin{array}{cc} & j \\ k & \left[\begin{array}{c} \cdots \frac{\partial y_k}{\partial x_j} \end{array} \right]_{m\times d} \end{array}
$$

→ What if my $\vec{x}, \vec{y}, \vec{z}$ are tensors?
  → string up into vectors & proceed
  → Matlab notation x-vec = X(:);
  → Trust me, this is the cleanest way

→ In Neural Nets $\vec{z} \in \mathbb{R}^1$ (Loss) $L(\vec{w})$



$$\vec{h}^{(i)} = g(\vec{h}^{(i-1)}, \vec{w})$$

$$L(\vec{w}) = f(h^{(i)})$$

$$\frac{\partial L}{\partial \vec{w}} = \frac{\partial L}{\partial \vec{h}^{(i)}} \frac{\partial \vec{h}^{(i)}}{\partial \vec{w}} \qquad \frac{\partial L}{\partial \vec{w}} = \left\langle \frac{\partial L}{\partial out}, \frac{\partial\, out}{\partial \vec{w}} \right\rangle$$

$$[\quad]_{1\times d} \; [\quad]_{1\times|h^i|} \; [\quad]_{|h^i|\times d}$$