Throughout the exam, the symbol \( \mathbb{N} \) denotes the natural numbers, and the symbol \( \mathbb{R} \) denotes the real numbers.

1. **Types.** Let \( B = \{-2, -1, 1, 2\} \), \( g : B \to B \) where \( f(x) = -x \). For each of the following expressions, indicate whether the type of the expression is set, proposition, function, or ill-formed:

   (a) \( B \subseteq \emptyset \)
   (b) \( B \times B \)
   (c) \( g \circ g \)
   (d) \( g \) is invertible
   (e) \( g = O(1) \)

2. **Big-Oh.** Let \( f \) and \( g \) be functions from \( \mathbb{N} \) to \( \mathbb{N} \) where \( f(n) = 2n^3 \) and \( g(n) = 1000n^2 \). Is \( f(n) = O(g(n)) \)? Prove that your answer is correct, using the logic-based definition of big-Oh.

3. **Big-Oh.** Let \( f_1, f_2 \) and \( g \) be functions from \( \mathbb{N} \) to \( \mathbb{N} \). Show that if \( f_1 \) is \( O(g) \) and \( f_2 \) is \( O(g) \) then \( f_1 + f_2 \) is \( O(g) \). (Note that \( (f_1 + f_2)(x) = f_1(x) + f_2(x) \)).

4. **Functions.** Consider the following informal description of a new function property:

   A function is a “foo function” if and only if every element in the codomain is mapped to by at least two elements in the domain.

   (a) Give a formal logic expression that is true exactly when the function \( f : S \to T \) is a foo function.
   (b) Give an example of a foo function, and prove that your function has the foo property.

5. **Functions.** Consider \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) given by \( f(a,b,c) = (c,a+b) \).

   (a) Is \( f \) one-to-one? If yes, give a proof; if no, give a counterexample.
   (b) Is \( f \) onto? If yes, give a proof; if no, give a counterexample.

6. **Sets.** Consider the proposition on sets \( A, B \) and \( C \).

   \[(A \cap B) \cup C = A \cap (B \cup C)\]

   (a) Give an example of sets \( A, B \) and \( C \) such that the proposition is false.
   (b) Give an example of sets \( A, B \) and \( C \) such that the proposition is true.
   (c) Give a condition that is necessary and sufficient for the proposition to be true. In other words, whenever the condition is true, the proposition is true; whenever the condition is false, the proposition is false.