Homework 2

Every assignment will be due at the beginning of class.

Constructing Proofs

Problem 1: Feedback (5 points)
Please answer after you finish the homework. How much time did you spend doing this homework? Which question(s) did you spend most time on?

Problem 2: Contapositive/Contradiction (30 points)
(a) Let \( x_1, \ldots, x_n \) be non-negative numbers and let \( \mu \) be their average, \( \mu = \frac{x_1 + \cdots + x_n}{n} \). Argued that at least one of the \( x_i \)'s is at least as large as \( \mu \).
(b) Argue that less than half of the \( x_i \)'s are larger than \( 2\mu \).
(c) Suppose that the \( x_i \)'s are people’s incomes and \( \mu \) is the average income. What does the statement that you proved in part (a) say about the number of people whose income is more than twice the average? More generally, what can you prove about the number of people whose income is more than \( k \) times the average?

Problem 3: Contrapositive/Contradiction (30 points)
Prove that, for all positive integers \( m \) of the form \( m = 4k + 3 \), \( \sqrt{m} \) is irrational.

Problem 4: True-False/Proof-Counterexample (5 points)
For each of the following propositions say if they are true or false. If true, give a proof. If false, give a counterexample.
(a) If \( m \) and \( n \) are non-zero integers, then \( m^2 + n^2 \) is never prime.
(b) If \( m \) and \( n \) are distinct non-zero integers, then \( m^2 - n^2 \) is never prime.
(c) If \( a \) and \( b \) are primes, then \( a^2 + b^2 \) is prime.

Problem 5: Basic Sums and Clever Representations (30 points)
Prove that, for all non-negative integers \( n \), \( 1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2 \).