Constructing Proofs

Problem 1: Feedback (5 points)
How much time do you spend every week for all the courses that you are taking? Include the time that you spend going to lectures, reading and doing your homeworks.

Problem 2: Simple Induction (30 points)
(a) Prove, by induction, that, for all non-negative integers \(n\), \(n^3 + (n+1)^3 + (n+2)^3\) is divisible by 9.
(b) Prove, by induction, that, for all non-negative integers \(n\), \(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}\).
(c) Prove, by induction, that, for all non-negative integers \(n\), \(8^n - 3^n\) is divisible by 5.

Problem 3: Induction and Algorithms (30 points)
An automotive teller machine has only 20 dollar bills and 50 dollar bills. Which amounts of money can the machine dispense, assuming that the machine has limitless supply of the two denominations of bills? Give an algorithm that the machine can use to determine how many bills of each denominations to output, when a client requests \(n \times 10\) dollars, for any \(n \geq 5\).

Problem 4: Induction and Recursion (35 points)
The Towers of Hanoi is a popular puzzle. It consists of 3 pegs, say A, B, and C, and a number of discs of differing diameters, each with a hole in the center. The discs initially sit on one of the pegs in order of decreasing diameter (smallest at top, largest at bottom), thus forming a triangular tower. The object is to move the tower to one of the other pegs by transferring the discs one at a time in such a way that no disc is ever placed upon a smaller one. In class we gave a recursive algorithm to solve this puzzle for \(n\) rings, \(n \geq 1\), and argued, inductively, that the algorithm is correct and uses \(2^n - 1\) moves. Suppose that we modify the traditional rules of the Towers of Hanoi by requiring that one move discs only to adjacent pegs. That is, we can move a disc from A to B, from C to B, and from B to either A or C. But we cannot move a disc from A to C, or from C to A. Give a recursive algorithm that solves this modification of the puzzle for \(n\) rings, \(n \geq 1\). Argue, inductively, that the algorithm is correct and uses at most \(3^n - 1\) moves.