Problem 1: 25 points
Which of the following propositions are true and which are false? For true propositions, give a short proof. For false propositions, give a counter-example.
(a) If \( n \) is an odd integer, then \( n^3 - 5 \) is an even integer.
(b) If \( n \) is an integer greater than 1, then \( n^2 - 1 \) is a prime number.
(c) There exist two consecutive integers greater than 2 that are prime, that is, for some integer \( p > 2 \), both \( p \) and \( p+1 \) are primes.

Problem 2: 25 points
Prove that \( \sqrt{7} \) is not rational.

Problem 3: 25 points
Prove that, for every integer \( n \geq 1 \), \( 7^n - 1 \) is divisible by 6.

Problem 4: 25 points
Prove that, for every integer \( n \geq 1 \), \( 1^3 + 2^3 + \ldots + n^3 = \frac{n(n+1)/2}{2^2} \).