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Problem 1: 25 points
Prove that, for all real numbers $x$, the quantity $3x^2 + 4x + 5$ is positive.

ANSWER:

\[
3x^2 + 4x + 5 = 2x^2 + (x^2 + 2 \cdot 2 \cdot x + 2^2) + 1 \\
= 2x^2 + (x + 2)^2 + 1
\]

which is obviously the sum of three quantities, the first two of which are non-negative and the third is positive, so the entire quantity is always positive.
Problem 2: 25 points

Prove that \( \sqrt{5} \) is not rational.

ANSWER:

Assume, for the purposes of contradiction, that \( \sqrt{5} \) is rational. Therefore,

\[
\sqrt{5} = \frac{p}{q},
\]

where \( p \) and \( q \) are relatively prime non-zero integers. Hence, by raising to the power of 2, we get that

\[
5q^2 = p^2 \quad (1).
\]

Now observe that both of \( p \) and \( q \) must be odd. (This is because, if they are both even they would not be relatively prime, and if one of them was even and the other one odd, the above equation would result in an even number on one side and an odd number on the other). So we may assume that, for some integers \( i \) and \( j \),

\[
\begin{align*}
p &= 2i + 1 & \text{and hence} & & p^2 &= 4i^2 + 4i + 1 \\
q &= 2j + 1 & \text{and hence} & & q^2 &= 4j^2 + 4j + 1
\end{align*}
\]

By substituting now to equation (1) we get that

\[
\begin{align*}
5(4j^2 + 4j + 1) &= 4i^2 + 4i + 1 & \text{hence} \\
20j^2 + 20j + 5 &= 4i^2 + 4i + 1 & \text{hence} \\
20j^2 + 20j + 4 &= 4i^2 + 4i & \text{hence} \\
5j^2 + 5j + 1 &= i^2 + i & \text{hence} \\
5j(j + 1) + 1 &= i(i + 1)
\end{align*}
\]

Now the quantities \( j(j + 1) \) and \( i(i + 1) \) are always even, because they are products of two consecutive integers and at least one of them (i.e., at least one of \( i \) and \( i + 1 \), and at least one of \( j \) and \( j + 1 \)) must be even. That makes the quantities \( 5j(j + 1) \) and \( i(i + 1) \) both even. Hence the left-hand-side of the last equality is odd, while the right-hand-side of the last equality is even. But this is impossible. Hence, our original assumption that \( \sqrt{5} \) is rational must be false.
Problem 3: 25 points

Prove that, for every integer \( n \geq 1 \), the quantity \( 9^n + 3 \) is divisible by 4.

ANSWER:

We will proceed by induction.

BASE CASE: When \( n = 1 \), the quantity become \( 9 + 3 = 12 \), which is obviously divisible by 4.

INDUCTIVE HYPOTHESIS: Assume that, for some \( k \geq 1 \), the quantity \( 9^k + 3 \) is divisible by 4.

INDUCTIVE STEP: We want to show that the quantity \( 9^{k+1} + 3 \) is divisible by 4. But,

\[
9^{k+1} + 3 = 9 \cdot 9^k + 3 \\
= (8 + 1) \cdot 9^k + 3 \\
= 8 \cdot 9^k + (9^k + 3)
\]

But the first part of the above quantity is a multiple of 8, so it is obviously divisible by 4, and the second part is a multiple of 4 by the inductive hypothesis.
Problem 4: 25 points

Chocolate often comes in rectangular bars marked off into small squares. It is easy to break a larger rectangle into two smaller rectangles along one of the horizontal or vertical lines between the squares. Suppose that I have a bar containing, originally, $n - 1$ horizontal and $n - 1$ vertical lines, thus $n^2$ squares, and wish to break it down into its individual squares. Prove that, no matter which way I break it, it will take exactly $n^2 - 1$ moves to do this.

**ANSWER: (MODUS PONENS)**

Each move increases the total number of pieces by exactly one. That means at most one and at least one.

Because each move increases the number of pieces by at most one, to start from one piece and end with $n^2$ pieces, we need at least $n^2 - 1$ moves.

Because each move increases the number of pieces by at least one, we can start from one piece and end with $n^2$ pieces in at most $n^2 - 1$ moves. Any algorithm will work:

- while there are less than $n^2$ pieces
  - take any piece larger than an individual square
  - break the piece in any legal way to two smaller pieces