Problem 1: Feedback (10 points)
Most people are happy with the lectures and found that the homework was of the right level of difficulty and helped them understand the material. A small handful found the material and homework too easy. If you fall in this latter category stop by my office after labor day and I can give you some extra project in discrete math or theoretical computer science to keep you busy and interested.

Problem 2: Direct Proof/Modus Ponens (30 points)
(a) Prove that, for all integers $n$, if $n$ is odd then $n^2$ is odd.
If $n$ is odd, then $n^2$ is the product of two odd numbers. But we know that the product of any two odd numbers is always odd.
(b) Prove that, for all integers $n$, $n^3 - n$ is divisible by 3.
For any integer $n$, $n^3 - n = n(n^2 - 1) = n(n-1)(n+1) = (n-1)n(n+1)$, which is the sum of three consecutive integers. But, among any three consecutive integers, one is necessarily a multiple of 3. So either $(n-1)$ or $n$ or $(n+1)$ is divisible by 3, so their product is also divisible by 3.
(c) Prove that, for all real numbers $x$, $2x^2 - 4x + 3 > 0$.
We will try to rewrite the above expression as a sum of positive quantities. Observe that $2x^2 - 4x + 3 = x^2 + x^2 - 2x - 2x + 2 + 1 = x^2 - 2x + 1 + x^2 - 2x + 1 + 1 = (x-1)^2 + (x-1)^2 + 1$. The latter expression is obviously always positive, since the terms $(x-1)^2$ are always non-negative and 1 is a positive number.

Problem 3: Proofs/Counter-Examples/Quantifiers (30 points)
(a) Is the following proposition true or false: ”For every prime $p$, $p+2$ is prime”. If true, give a proof. If false, give a counter-example.
The proposition is false. Here’s a counter-example. For $p=7$, which is a prime number, the number $p+2=9=3^2$ is not prime.
(b) Is the following proposition true or false: ”There exist three consecutive odd integers that are primes, that is, for some odd number $p$, $p$, $p+2$, and $p+4$ are primes”. If true, give a proof. If false, give a counter-example.
The proposition is true. In particular, for $p=3$, which is a prime number, the integers $p+2=5$ and $p+4=7$ are also prime numbers.
(c) Is the following proposition true or false: ”For every pair of (not necessarily distinct) irrational numbers $x$ and $y$, the product $x \cdot y$ is irrational”. If true, give a proof. If false, give a counter-example.
The proposition is false. Here’s a counter-example. For $x = y = \sqrt{2}$, The product $x \cdot y = \sqrt{2} \cdot \sqrt{2} = 2$ is not irrational.

Problem 4: Puzzle (Modus Ponens) (30 points)
(a) Prove that an $8 \times 8$ chessboard can be completely covered using dominos ($2 \times 1$ pieces).
For each line put 4 dominos horizontally. In this way you can cover the chessboard with 32 dominos.
(b) Prove that it is impossible to cover completely with dominos the $8 \times 8$ chessboard with one square at a corner of the board removed.
If we remove one square from the chessboard we need to cover 63 squares. On the other hand, each domino covers 2 squares, therefore, for any $k$, a tiling that uses $k$ dominos will cover $2k$ squares, which is an even number of squares.
(c)* Prove that it is impossible to cover completely with dominos the $8 \times 8$ chessboard with two squares at opposite corners of the board removed.
Let us color the squares of the chessboard black and white in the usual way. Now we notice two things. First, that the squares of two opposite corners of the $8 \times 8$ chessboard are of the same color. So removing two opposite squares will leave us with 30 black and 32 white squares. The second thing to notice is that each $2 \times 1$ domino, whether placed horizontally or vertically, covers two neighboring squares, which must necessarily be one black and one white. Hence any tiling with $2 \times 1$ dominos must necessarily cover the same number of black and white squares.