Problem 1: Feedback (10 points)
How did you feel about the first quiz?

Problem 2: Induction (20 points)
(a) Prove that, for all integers \( n \geq 1 \), \( 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \).

Answer: Prove by induction.
Basecase: for \( n = 1 \), \( 1^3 = 1 = \frac{1^2(1+1)^2}{4} \).
Inductive hypothesis: Assume that for \( n = k \), \( 1^3 + 2^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4} \)
Inductive step: We want to show that, for \( n = k+1 \), \( 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4} \).
And,
\[
1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 \text{by the inductive hypothesis}
\]
\[
= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}
\]
\[
= \frac{(k^2 + 4(k+1))(k+1)^2}{4}
\]
\[
= \frac{(k+1)^2(k+2)^2}{4}
\]

(b) Prove that, for all integers \( n \geq 1 \), \( 1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2 \).

Answer: Use previous result from problem a), we have \( 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \), and we know that \( 1+2+\ldots+n = \frac{n(n+1)}{2} \). Therefore \( 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = (1+2+\ldots+n)^2 \).

Problem 3: Induction (20 points)
Show that, for every integer \( n \geq 1 \), and for any integers \( a \) and \( b \) with \( a \neq b \), \( a^n - b^n \) is divisible by \( a - b \).

Answer: Prove by strong induction.
Basecase: for \( n = 1 \), \( a^1 - b^1 = a - b \) is divisible by \( a - b \), for \( n = 2 \), \( a^2 - b^2 = (a + b)(a - b) \) is also divisible by \( a - b \).
Inductive hypothesis: Let \( k \) be an integer and \( k \geq 2 \). Assume that for all \( 1 \leq m \leq k \), \( a^m - b^m \) is divisible by \( a - b \).
Inductive step: We need to show that for \( n = k+1 \), \( a^{k+1} - b^{k+1} \) is divisible by \( a - b \). And,
\[
a^{k+1} - b^{k+1} = a \cdot a^k + b \cdot b^k
\]
\[
= (a - b + b)a^k - (b - a + a)b^k
\]
\[
= (a - b)a^k + ba^k - (b - a)b^k - ab^k
\]
\[
= (a - b)(a^k - b^k) + ba^k - ab^k
\]
\[
= (a - b)(a^k - b^k) + ab(a^{k-1} - b^{k-1})
\]
By assumption, $a^k - b^k$ and $a^{k-1} - b^{k-1}$ are divisible by $a - b$, therefore $a^{k+1} - b^{k+1}$ is divisible by $a - b$.

**Problem 4: Induction, Recursion and Algorithms (50 points)**

(a) Given an equal arm balance capable of determining only the relative weights of two quantities, and eight coins, all of equal weight except possibly one which is lighter, explain how to determine if there is a light coin and how to identify it in three weighings.

**Answer:** Split the eight coins into two halves and put four coins on each side of the balance. If one side is lighter, there is a light coin. Then split those four coins into two halves and weigh them. Determine the lighter side. Then again weigh the two coins on the lighter side on the balance. The lighter one is the the one we are looking for.

(b) Given an equal arm balance capable of determining only the relative weights of two quantities, and $2^n$ coins, $n \geq 1$, all of equal weight except possibly one which is lighter, explain how to determine if there is a light coin and how to identify it in $n$ weighings.

**Answer:** Prove by induction.

**Base case:** for $n = 1$, we just put one coin on each side of the balance, if one side is lighter, there is a lighter coin and the coin on that side is the one.

**Inductive hypothesis:** assume for $n = k$, given $2^k$ coins, we can determine if there is a light coin and can identify it in $k$ weighings.

**Inductive step:** We need to show that given $2^{k+1}$ coins, we can determine if there is a light coin and can identify it in $k+1$ weighings.

Split the coins into two halves and put $2^k$ coins on each side of the balance. If one side is lighter, there is a light coin and the coin is on that side. By assumption, for those $2^k$ coins we can identify the light coin in $k$ with $k$ weighing. Therefore, in total, we can determine if there is a light coin and can identify it in $k+1$ weighing.

The algorithm works as recursively splitting the coins into two halves and weighing those two halves.

(c) Given an equal arm balance capable of determining only the relative weights of two quantities, and eight coins, all of equal weight except possibly one which is lighter, explain how to determine if there is a light coin and how to identify it in just two weighings.

**Answer:** Split the coins into three groups, say A, B and C, with three coins in group A and B and two coins in group C.

Weigh A against B. If one group is lighter, we know there is a light coin and the coin is in that three coins. Then pick two of the three coins and weigh them. If there one of them is lighter, we find the light coin. Otherwise, the third coin is the light coin.

If group A and B weigh equal, we then weigh the two coins in group C. If one of them is lighter, we then identify the light coin. Otherwise, we know there is no light coin in the eight coins.

(d) Given an equal arm balance capable of determining only the relative weights of two quantities, and $3^n - 1$ coins, all of equal weight except possibly one which is lighter, explain how to determine if there is a light coin and how to identify it in $n$ weighings.

**Answer:** Prove by induction.

**Base case:** for $n = 2$, we have just proved in problem c) that we can determine if there is a light coin and identify it in 2 weighing.

**Inductive hypothesis:** assume for $n = k$, given $3^k - 1$ coins, we can determine if there is a light coin and can identify it in $k$ weighing.
Inductive step: We need to show that given $3^{k+1} - 1$ coins, we can determine if there is a light coin and can identify it in $k+1$ weighing.

Split the coins into three groups of coins, with $3^k$ coins in group A and B and $3^k - 1$ coins in group C.

Weigh A against B. If one group is lighter, we know there is a light coin and the coin is in that group. Take one coin out from that group and we have $3^k - 1$ coins. By assumption we can determine if there is a light coin in that $3^k - 1$ coins and can identify it in $k$ weighing.

If there is no light coin in that $3^k - 1$ coins, we know that the one coin we take out is the light one.

If group A and B weigh equal, we then weigh the $3^k - 1$ coins in group C. By assumption, we can determine if there is a light coin in that $3^k - 1$ coins and can identify it in $k$ weighing.

Therefore, in total, we can determine if there is a light coin and can identify it in $k+1$ weighing.

The algorithm works as recursively splitting the coins into three groups of coins and weight them as described above.