Problem 1: 25 points
Which of the following propositions are true and which are false? For true propositions, give a short proof. For false propositions, give a counter-example.
(a) If \( n \) is an odd integer, then \( n^3 - 5 \) is an even integer.
Answer: True. Because the product of two odd integers is always odd, \( n^3 = (n \times n) \times n \) is odd.
And the sum of two odd integers is even, therefore \( n^3 - 5 \) is an even integer.
(b) If \( n \) is an integer greater than 1, then \( n^2 - 1 \) is a prime number.
Answer: False. For \( n = 3 \), \( n^2 - 1 = 3^2 - 1 = 8 \) is not a prime number.
(c) There exist two consecutive integers greater than 2 that are prime, that is, for some integer \( p > 2 \), both \( p \) and \( p+1 \) are primes.
Answer: False. For any two consecutive integers, one must be even. And because it is greater than 2, it must have a factor of 2, therefore it is not a prime.

Problem 2: 25 points
Prove that \( \sqrt{7} \) is not rational.
Answer: Prove by contradiction.
For the purpose of contradiction, assume \( \sqrt{7} = \frac{p}{q} \), for some \( p \), \( q \) and \( p \) and \( q \) have no common factors. Then.
\[
7 = \frac{p^2}{q^2} \text{ and hence } 7 \cdot q^2 = p^2
\]
Now notice that both \( p \) and \( q \) must be odd, otherwise either they have common factor 2 or one side of the equation is odd and the other is even. Suppose therefore that \( p = 2i + 1 \) and \( q = 2j + 1 \), and by substituting to the first equation we get
\[
7 \cdot (2i + 1)^2 = (2i + 1)^2
\]
\[
7 \cdot (4i^2 + 4i + 1) = 4i^2 + 4i + 1
\]
\[
4 \cdot 7 \cdot (j^2 + j) + 7 = 4i^2 + 4i + 1
\]
\[
4 \cdot 7 \cdot (j^2 + j) + 6 = 4i^2 + 4i
\]
\[
2 \cdot 7 \cdot (j^2 + j) + 3 = 2i^2 + 2i
\]
However, the left side of the equation is odd and the right side of the equation is even, which can be true. Therefore, the assumption that \( \sqrt{7} \) is rational must be false.

Problem 3: 25 points
Prove that, for every integer \( n \geq 1 \), \( 7^n - 1 \) is divisible by 6.
Answer, First Way: Prove by induction.
Base case: for \( n = 1 \), \( 7^1 - 1 = 6 \), obviously divisible by 6.
Inductive hypothesis: Assume that for \( n = k \geq 1 \), \( 7^k - 1 \) is divisible by 6.
Inductive step: We want to show that, for \( n = k+1 \geq 1 \), \( 7^{k+1} - 1 \) is divisible by 6. And,
\[
7^{k+1} - 1 = 7 \cdot 7^k - 1
\]
\[
= 7(7^k - 1) + 7 - 1
\]
\[
= 7(7^k - 1) + 6
\]
By assumption $7^k - 1$ is divisible by 6 and 6 is obviously divisible by 6, therefore $7^{k+1} - 1$ is divisible by 6.

Answer, Second Way: We know, from class, that

$$7^{n-1} + \ldots + 7 + 1 = \frac{7^n - 1}{7 - 1} = \frac{7^n - 1}{6}$$

But the left hand side of the above equation is an integer, so the right hand side must be also an integer. So the numerator $7^n - 1$ must be divisible by the denominator 6.

Problem 4: 25 points

In class we showed that every connected graph with all its degrees even has an Euler tour, that is, a way to start from a vertex $v$, traverse each edge of the graph exactly once and return to $v$, without raising the pen from the paper.

Now let us consider a connected graph with $n \geq 2$ vertices where all its degrees are even, except the degrees of exactly two vertices, namely $u$ and $v$ which are odd. Argue that for every such graph, there is a way to start from vertex $v$, traverse each edge of the graph exactly once and end up in vertex $u$, without raising the pen from the paper.

Answer: Let’s add an edge $(u, v)$ to the original graph and get a new graph $G’$. Now the degree of all the vertices of $G’$ are even. So it has an Euler tour, that is, a way to start from vertex $u$, traverse each edge of the graph exactly once and return to $u$. Suppose the entire path of this tour is

$$(u, p_1), (p_1, p_2), \ldots, (p_i, u), (u, v), (v, p_j), \ldots, (p_{(n-1)}, p_n), (p_n, u)$$

Now we can just remove the edge $(u, v)$ from the above path to construct a new path that starts from vertex $v$, traverses each edge of the original graph exactly once and ends up in vertex $u$, that is,

$$(v, p_j), \ldots, (p_{(n-1)}, p_n), (p_n, u), (u, p_1), (p_1, p_2), \ldots, (p_i, u)$$