**Problem 2:** Argue that every \( n \) element set has \( \frac{n(n-1)}{2} \) distinct two-element subsets.

**Proof.** Base Case - True for \( n = 1 \) as it will have 0 = \( \frac{1(1-1)}{2} \) subsets with two elements. True for \( n = 2 \) as it will have 1 = \( \frac{2(2-1)}{2} \) subset with two elements. 

Inductive Hypothesis - Assume that every \( n = k-1 \) element set has \( \frac{(k-1)(k-2)}{2} \) distinct two-element subsets.

Inductive Step - \( n = k \). When we add \( k^{th} \) element, the set will have all the two-element subsets of the set with \( k-1 \) elements plus it will have the new subsets with \( k^{th} \) with every other \( (k-1) \) elements in the set. So, total number of subsets will be \( \frac{(k-1)(k-2)}{2} + (k-1) = \frac{k^2-3k+2+2k-2}{2} = \frac{k^2-k}{2} = \frac{k(k-1)}{2} \).

**Problem 3:**

(a) Show that for every integer \( x \geq 3 \) and for every integer \( n \geq 2 \), the integer \( x^n - 1 \) is composite.

**Proof.** First, we show that \( x^n - 1 \) is divisible by \( x - 1 \) for all \( n \geq 2 \).

Base Case - \( n = 2 \) \( \Rightarrow \) \( x^2 - 1 = (x - 1)(x + 1) \). Therefore, \( x^2 - 1 \) is divisible by \( x - 1 \).

Inductive Hypothesis - True for \( n = k - 1 \) i.e. \( x^{k-1} - 1 \) is divisible by \( x - 1 \), \( x^{k-1} - 1 = p(x - 1) \).

Inductive Step - \( n = k \)

\[
x^k - 1 = x \cdot x^{k-1} - 1 \\
= (p(x - 1)) + (x - 1) \\
= px(x - 1)(x - 1)
\]

Therefore, \( x^k - 1 \) is divisible by \( x - 1 \).

Since \( x \geq 3 \), \( x - 1 \geq 2 \). So \( x^n - 1 \) has a factor greater than 1 and smaller than \( x^n - 1 \) and hence is a composite number. \( \square \)
(b) Show that for every integer \( n \geq 1 \) and for any integers \( a \) and \( b \) with \( a \neq b \), \( a^n - b^n \) is divisible by \( a - b \).

**Proof.**

**Base Case** - \( n = 1 \) \( \Rightarrow a - b \) is divisible by \( a - b \).

**Inductive Hypothesis** - True for \( n = k - 1 \) i.e. \( a^{k-1} - b^{k-1} \) is divisible by \( a - b \), \( a^{k-1} - b^{k-1} = p(a - b) \).

**Inductive Step** - \( n = k \)

\[
\begin{align*}
a^k - b^k &= a \cdot a^{k-1} - b \cdot b^{k-1} \\
&= a(a^{k-1} - b^{k-1}) + a \cdot b^{k-1} - b \cdot b^{k-1} \\
&= a(p(a - b)) + b^{k-1}(a - b) \\
&= (pa + b^{k-1})(a - b)
\end{align*}
\]

Therefore, \( a^n - b^n \) is divisible by \( a - b \).  \( \square \)

**Problem 4:**

(a) Solve the recurrence \( T(n) = T(n/2) + 1 \), when \( T(1) = 1 \) and \( n \) is a power of 2.

Since \( n \) is a power of 2, let \( n = 2^k \).

\[
T(n) = T\left(\frac{n}{2}\right) + 1 \\
= T\left(\frac{n}{2^2}\right) + 2 \\
= T\left(\frac{n}{2^3}\right) + 3 \\
\vdots \\
= T\left(\frac{n}{2^k}\right) + k \\
= T(1) + k \\
= \log_2(n) + 1
\]

(b) Solve the recurrence \( T(n) = T(n/2) + n \), when \( T(1) = 1 \) and \( n \) is a power of 2

Since \( n \) is a power of 2, let \( n = 2^k \).
\[ T(n) = T\left(\frac{n}{2}\right) + n \]
\[ = T\left(\frac{n}{2^2}\right) + 2n \]
\[ = T\left(\frac{n}{2^3}\right) + 3n \]
\[ = \ldots \]
\[ = T\left(\frac{n}{2^k}\right) + kn \]
\[ = T(1) + k \cdot 2^k \]
\[ = n \log_2(n) + 1 \]

Problem 5:

(a) Divide 8 coins to two sets of 4 each. Weigh (first weighing) the two sets on the two sides of the equal arm balance. If one of the two sets is lighter then 1 of the coins in that set is lighter than other 7. Choose that set and divide 4 coins into two sets of 2 each. Weigh (second weighing) the two sets on the two sides of the equal arm balance. One of the two sets will be lighter. Choose that set. Divide 2 coins into two sets of 1 each. Weigh (third weighing) the two sets on the two sides of the equal arm balance. Thus, the lighter coin can be identified in 2 weighings.

(b) (By Induction)

Base Case - True for \( n = 1 \). There are \( 2^1 \) coins. Weigh one coin on each side of the two sides of the equal arm balance. Thus the lighter coin can be identified in 1 weighing.

Inductive Hypothesis - Assume for \( n = k - 1 \) i.e. the lighter coin out of \( 2^{k-1} \) coins can be identified in \( k - 1 \) weighings.

Inductive Step - Divide \( 2^k \) coins into two sets of \( 2^{k-1} \) each. Weigh the two sets on the two sides of the equal arm balance. If one of the two sets is lighter then 1 of the coins in that set is lighter than others. Choose that set of \( 2^{k-1} \) coins. The lighter coin can be identified from this set in \( k - 1 \) weighings. Therefore, total \( k \) weighings required.

(c) Divide 8 coins into three sets of 3, 3, 2 coins. Weigh (first weighing) two sets of 3 coins on the two sides of the equal arm balance. These two sets can be of the same weight or one can be lighter than the other.
Case 1 - If one of the two sets is lighter then 1 of the coins in that set is lighter than other 7. Divide 3 coins into three sets of 1 each. Weigh (second weighing) two sets of 1 coin on the two sides of the equal arm. If one of two sets is lighter than the other than that coin is the lighter coin. Otherwise the third coin of this set is lighter.

Case 2 - If the two sets are of equal weight then take the third set of 2 coins. Divide 2 coins into two sets of 1 each. Weigh (second weighing) these sets of 1 coin on the two sides of the equal arm. If one of two sets is lighter than the other than that coin is the lighter coin. Thus, the lighter coin can be identified in 2 weighings.

(d) (By Induction) Base Case - True for $n = 1$. There are $3^1 - 1 = 2$ coins. Weigh one coin on each side of the two sides of the equal arm balance. Thus the lighter coin can be identified in 1 weighing.

Inductive Hypothesis - Assume for $n = k – 1$ i.e. the lighter coin out of $3^{k-1} - 1$ coins can be identified in $k – 1$ weighings.

Inductive Step - $3^k - 1 = 3^{k-1} + 3^{k-1} + (3^{k-1} – 1)$. Divide $3^k - 1$ coins into three sets of $3^{k-1}$, $3^{k-1}$ and $3^{k-1} – 1$ coins. Weigh the two sets of $3^{k-1}$ coins on the two sides of the equal arm balance. If one of the two sets is lighter then 1 of the coins in that set is lighter than others. Otherwise the set that was not weighed has the lighter coin. Choose the appropriate set of $3^{k-1}$ or $3^{k-1} – 1$ coins. The lighter coin can be identified from this set in $k – 1$ weighings. Therefore, total $k$ weighings required.