Problem 2:

(a) Chebyshev’s Inequality - \( \Pr\{|X - E[X]| \geq \epsilon \} \leq \frac{\Var[X]}{\epsilon^2} \)

(b) \( \Pr\{|X < 49000 \text{ or } X > 51000\} \)
\[ = \Pr\{|X - 50000 < -1000 \text{ or } X - 50000 > 1000\} \]
\[ = \Pr\{|X - 50000| > 1000\} \]
\[ = \Pr\{|X - E[X]| > 1000\} \]
\[ \leq \frac{\Var[X]}{(1000)^2} \text{ (By Chebychev’s Inequality)} \]
\[ = \frac{2500}{1000^2} = 0.0025 \]
Therefore, \( \Pr\{|X < 49000 \text{ or } X > 51000\} \leq 0.0025 \).

\( \Pr\{49000 < X < 51000\} \)
\[ = 1 - \Pr\{|X < 49000 \text{ or } X > 51000\} \]
\[ \geq 1 - 0.0025 = 0.9975 \]
Therefore, \( \Pr\{49000 < X < 51000\} \geq 0.9975 \).

Problem 3: To apply Chebyshev’s Inequality, we need \( E[Z] \) and \( \Var[Z] \). Since \( Z \) is the difference between number of heads (\( X \)) and number of tails (\( Y \)), we can model \( Z \) as follows:
For \( 1 \leq i \leq n \), let \( Z_i \) be a random variable defined as
\[ Z_i = \begin{cases} 
1 & \text{if } i^{th} \text{ flip is Heads} \\
-1 & \text{if } i^{th} \text{ flip is Tails}
\end{cases} \]
Given this definition of \( Z_i \)’s, \( Z = \sum_{i=1}^{n} Z_i \).
We have,
\[ E[Z_i] = (1)(1/2) + (-1)(1/2) = 0 \]
Therefore,
\[ E[Z] = E[\sum_{i=1}^{n} Z_i] = \sum_{i=1}^{n} E[Z_i] = 0 \]
Now, we need

\[ Var(Z_i) = E[Z_i^2] - (E[Z_i])^2 \]
\[ = \left\{ (1)^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} \right\} - \{0\} \]
\[ = 1 \]

For the variance of \( Z \), we use the fact that \( Z_i \)'s are independent events. Outcome of any flip does not depend on any other flip. Therefore,

\[ Var(Z) = Var(\sum_{i=1}^{n} Z_i) = \sum_{i=1}^{n} Var(Z_i) = \sum_{i=1}^{n} 1 = n \]

Therefore, by Chebyshev’s inequality,

\[ Pr\{|Z - E[Z]| > 10\sqrt{n}\} \leq \frac{Var(Z)}{(10\sqrt{n})^2} \]
\[ \Rightarrow Pr\{|Z| > 10\sqrt{n}\} \leq \frac{n}{100n} = \frac{1}{100} = 0.01 \]
\[ \Rightarrow Pr\{Z > 10\sqrt{n}\} \leq 0.01 \]

**Problem 4:** Expected Score = Expected score on true/false questions + Expected score on multiple choice questions

Expected score on ONE true/false questions = (2 \times \text{Probability of being correct}) + (0 \times \text{Probability of being wrong}) = 2(0.9) + 0(1-0.9) = 1.8

Expected score on 50 true/false questions = 50 \times 1.8 = 90

Expected score on ONE multiple choice questions = (4 \times \text{Probability of being correct}) + (0 \times \text{Probability of being wrong}) = 4(0.8) + 0(1-0.8) = 3.2

Expected score on 25 true/false questions = 25 \times 3.2 = 80

Therefore, Expected Score = 90 + 80 = 170

**Problem 5:**

(a) Since there are 5 “true’s”, we need to choose 5 positions out of 20 where “true” needs to be placed. In the rest of the places, “false” will be put. Hence number of possible keys is \( \binom{20}{5} \).

(b) Expected number of “true’s” that the student gets right = Number of “true’s” \times \text{Probability of assigning “true”} = 5 \times \frac{1}{2} = 2.5

Expected number of “false’s” that the student gets right = Number of
“false’s” × Probability of assigning “false” = 15 × $\frac{1}{2} = 7.5$
Therefore, Expected number of questions that the student answers correctly = 2.5 + 7.5 = 10.

(c) Expected number of “true’s” that the student gets right = Number of “true’s” × Probability of assigning “true” = 5 × $\frac{1}{4} = 1.25$
Expected number of “false’s” that the student gets right = Number of “false’s” × Probability of assigning “false” = 15 × $\frac{3}{4} = 11.25$
Therefore, Expected number of questions that the student answers correctly = 1.25 + 11.25 = 12.5.