Chapter 18
On the Number of Eulerian Orientations of a Graph

Milena Mihail* Peter Winkler*

Abstract

We give efficient randomized schemes to sample and approximately count Eulerian orientations of any Eulerian graph. Eulerian orientations are natural flow-like structures, and Welsh has pointed out that computing their number **(i)** corresponds to evaluating the Tutte polynomial at the point $(0, -2)$ [8,19] and **(ii)** is equivalent to evaluating "ice-type partition functions" in statistical physics [20].

Our algorithms are based on a reduction to sampling and approximately counting perfect matchings for a class of graphs for which the methods of Broder [3,10] and others [4,6] apply. A crucial step of the reduction is the "Monotonicity Lemma" (Lemma 3.3) which is of independent combinatorial interest. Roughly speaking, the Monotonicity Lemma establishes the intuitive fact that "increasing the number of constraints applied on a flow problem can only decrease the number of solutions". In turn, the proof of the lemma involves a new decomposition technique which decouples problematically overlapping structures (a recurrent obstacle in handling large combinatorial populations) and allows detailed enumeration arguments. As a byproduct, **(i)** we exhibit a class of graphs for which perfect and near-perfect matchings are polynomially related, and hence the permanent can be approximated, for reasons other than "short augmenting paths" (previously the only known approach); and **(ii)** we obtain a further direct sampling scheme for Eulerian orientations which is faster than the one suggested by the reduction to perfect matchings.

Finally, with respect to our approximate counting algorithm, we give the complementary hardness result, namely, that counting exactly Eulerian orientations is $\#P$-complete, and provide some connections with Eulerian tours.

1 Introduction

Consider an undirected Eulerian graph, that is, a graph in which all vertices have even degree. An *Eulerian orientation* of the graph is an orientation of its edges such that for every vertex v the number of edges directed towards v is equal to the number of edges directed out of v: $d_{in}(v) = d_{out}(v)$. It is well known that finding an Eulerian orientation can be accomplished efficiently; in this paper we are concerned with the questions of *sampling* (i.e. generation from a distribution arbitrarily close to uniform) and *counting* the set of Eulerian orientations of an arbitrary Eulerian graph.

The significance of counting the number of Eulerian orientations was raised by Welsh in the context of examining the computational complexity of some problems in statistical physics [20]. In particular, Welsh observed that in the so-called "ice-type model", the crucial partition function "Z_{ICE}" is equal to the number of Eulerian orientations of some underlying Eulerian graph (for further details see also [2,14]). Apparently, no reasonable algorithmic or hardness result was known for evaluating Z_{ICE}, even for 4-regular planar graphs. Thus, answering one of Welsh's questions, our approximate counting scheme for Eulerian orientations (Theorem 3.4) is the first efficient scheme to approximate Z_{ICE}, while our $\#P$-completeness result (Theorem 5.1) explains the lack of any formula or other efficient method to compute Z_{ICE} exactly. (The list of Welsh's problems contains the monomer-dimer problem which translates directly to perfect matchings and has been treated in [3,9,10,18]; the Ising model, which has been treated in [12]; the ice-type model which is treated here; and problems involving percolation theory, polyominoes, and self-avoiding walks, which remain open.)

In addition, it has been also observed [8,19] that the counting problem for Eulerian orientations corresponds to evaluating the Tutte polynomial at the point $(0, -2)$, therefore the approximation and hardness results presented here have a direct translation on the map of the Tutte plane. Other fundamental problems that correspond to points or curves on the Tutte plane are, for example, spanning trees and forests, colorings, acyclic orientations, reliability, the Ising model, etc.

From one further practical point of view, if a graph represents the topology of a network, an Eulerian orientation is a uni-directional configuration of the network that preserves flows through nodes, thus, a maximum

*Bell Communications Research, 445 South Street, Morristown, NJ 07960.
mihail@flash.bellcore.com pw@flash.bellcore.com

global flow without sources and sinks. Consequently sampling Eulerian orientations amounts to observing a random instance of such a network, while counting the number of Eulerian orientations is equivalent to counting the number of maximum flows around the network. Presumably, networks with larger total number of maximum flows are better behaved.

From a theoretical point of view, the question of sampling and counting Eulerian orientations falls in the category of problems for which constructing a solution is in polynomial time, but the apparently harder problems of sampling and counting the set of all solutions, even in an approximation sense, are either significantly more involved, or intractable, unless unexpected collapses of complexity classes occur. Perhaps the most outstanding paradigm of such behavior, which is also the one that we shall use for both our algorithmic and hardness results, concerns the case of perfect matchings of bipartite graphs: it is well known that a perfect matching can be constructed in polynomial time, but counting the number of perfect matchings was shown in Valiant's seminal paper on permanents to be complete for the class $\#P$ [18]. In turn, the latter hardness result directed efforts towards obtaining efficient approximations.

The equivalence of randomized approximate counting and sampling was established by Jerrum, Valiant, and Vazirani [13], followed by the concrete scheme of Broder [3] and Jerrum and Sinclair [10] for approximately counting perfect matchings using the novel Markov chain simulation technique for sampling (see also [4,6] for various improvements, and [7,11,12,15] for further relevant results). More specifically, the known results for perfect matchings suggest that sampling and approximate counting can be achieved via randomized schemes which run in time polynomial in the size of the input graph, the inverse of the desired approximation accuracy, and the ratio of the total numbers of near-perfect over perfect matchings $|\mathcal{M}_{n-1}|/|\mathcal{M}_n|$ of the input graph (and with exponentially small failure probabilities, as usual).

In Section 3 we show that sampling and approximately counting Eulerian orientations reduces to sampling and approximately counting perfect matchings for a class of graphs for which $|\mathcal{M}_{n-1}|/|\mathcal{M}_n| = \mathcal{O}(n^4)$, thus we obtain efficient solutions for Eulerian orientations. It is interesting to remark that all known bounds on $|\mathcal{M}_{n-1}|/|\mathcal{M}_n|$ for classes of graphs that were treated previously (dense graphs [3], random and expander graphs [10], graphs with large factors [4], graphs arising from degree sequences [11] etc.) were obtained by showing that for any near-perfect matching there is a "short" augmenting path (e.g. constant length for dense graphs, $\log n$ length for expander graphs etc.) Thus any perfect matching can be associated with only polynomially few near-perfect matchings, which yields the bound. In contrast, our bound on $|\mathcal{M}_{n-1}|/|\mathcal{M}_n|$ is not based on a short augmenting path argument.

Roughly, the reason why $|\mathcal{M}_{n-1}|$ is not much larger than $|\mathcal{M}_n|$ for the class of graphs that arise in our reduction, is that these cardinalities are related to the cardinalities of certain sets of orientations, hence solutions to flow-like problems, which exhibit particularly favorable behavior: "increasing the number of constraints in a flow problem can only decrease the number of solutions". For example, the simplest case is when two vertices u and v are connected with $2d$ parallel edges. There are $\binom{2d}{d}$ ways to orient the edges so that both u and v have the same number of incoming and outgoing edges, but there are only $\binom{2d}{d-1}$ ways to orient the edges so that u has two more outgoing than incoming edges, while v has two more incoming than outgoing edges. In Monotonicity Lemma 3.3 we show that, in some sense, locally, all graphs behave like two vertices connected with parallel edges. The difficulty in establishing such a fact is that in arbitrary graphs we cannot isolate disjoint paths between any pair of vertices. To bypass this problem, we introduce a technique of decomposing graphs into "Euler elements" which decouples problematically overlapping paths.

Decoupling problematically overlapping structures is a recurrent obstacle in handling cardinalities related to large combinatorial populations. For example, one way to establish the crucial "rapid mixing" property for Markov chains defined on such populations is to consider a substantial number of paths between states of the Markov chain, and argue that no transition edge of the chain is congested with too many paths (hence the underlying graph of the Markov chain has no "small" cuts, and a random walk on such an "expander" graph converges fast; see [10] for an introduction to this reasoning, and [6,4,12] for further applications and improvements). Now the difficulty in such arguments is to decouple and bound the number of paths that may use the same transition edge. In Section 4 we define a natural Markov chain $\mathcal{MC}$ on orientations of Eulerian graphs and outline how our technique of decomposition into Euler elements facilitates the choice of random paths without congestion. Hence, $\mathcal{MC}$ is rapidly mixing. Using directly the simulation of $\mathcal{MC}$ as an efficient sampling scheme for Eulerian orientations saves a polynomial factor over the indirect reduction to matchings.

In Section 5 we show that counting Eulerian orientations exactly is $\#P$-complete, thus justifying the necessity of our efficient scheme for approximate counting, as well as the lack of any computationally reasonable exact counting method. By one further application of decompositions into Euler elements we observe a connection with numbers of Eulerian tours.

The rest of the paper is organized as follows: Section 2 reviews the known results for matchings, and introduces some terms for the treatment of orientations. In Section 3 we give the reduction from sampling and approximately counting Eulerian orientations to sampling and approximately counting perfect matchings for a class of graphs that cen be handled efficiently. In Section 4 we outline how the decomposition technique can yield the rapid mixing property for a Markov chain defined directly on orientations, and hence a faster sampling scheme. We discuss the potential of such an approach to yield faster approximate counting algorithms. Section 5 contains the complementary hardness result and a connection with numbers of Euler tours. Summary and open problems are in Section 6.

2 Preliminaries

For some set $\mathcal{S} \neq \emptyset$, a *d-sampling scheme* for $\mathcal{S}$ is an algorithm that, with probability at least $1-\delta$, outputs a $s \in \mathcal{S}$ such that

$$\sum_{x \in \mathcal{S}} \left| \Pr[s=x] - \frac{1}{|\mathcal{S}|} \right| < d \quad .$$

An (ϵ, δ)*-approximation scheme* for $\mathcal{S}$ is an algorithm that outputs an estimate $|\tilde{\mathcal{S}}|$ for $|\mathcal{S}|$ such that

$$\Pr\left[\big| |\mathcal{S}| - |\tilde{\mathcal{S}}| \big| \leq \epsilon |\mathcal{S}| \right] \geq 1-\delta \quad .$$

Our aim is to obtain *efficient* d-sampling and (ϵ,δ)-approximation schemes for the set $\mathcal{P}_0$ of Eulerian orientations of any Eulerian graph $G=(V,E)$, where $|V|=n$. By efficient, we mean running times polynomial in n, ϵ^{-1}, $\log d^{-1}$, and $\log \delta^{-1}$, as usual.

Let $G' = (V_1', V_2', E')$ be a bipartite graph, where $|V_1'| = |V_2'| = 2n'$. Let $\mathcal{M}_{n'}$ and $\mathcal{M}_{n'-1}$ be the sets of perfect matchings and near-perfect matchings of G' respectively (a near-perfect matching is a matching of size $n'-1$).

FACT 2.1. **[Broder [3], Jerrum and Sinclair [10], improvements in [6].]** *For any graph G' as above, there is a d-sampling scheme for the set of perfect matchings $\mathcal{M}_{n'}$. The scheme runs in time polynomial in n', $\log d^{-1}$, $\log\delta^{-1}$, and the ratio $|\mathcal{M}_{n'-1}|/|\mathcal{M}_{n'}|$.*

FACT 2.2. **[Broder [?] , Jerrum and Sinclair [10], improvements in [4,6].]** *For any graph G' as above, there is an (ϵ,δ)-approximation scheme for the set of perfect matchings $\mathcal{M}_{n'}$. The scheme runs in time polynomial in n', ϵ^{-1}, $\log\delta^{-1}$, and the ratio $|\mathcal{M}_{n'-1}|/|\mathcal{M}_{n'}|$.*

In Section 3 we show that sampling and approximate counting of Eulerian orientations is polynomial-time reducible to sampling and approximately counting perfect matchings for a class of graphs for which the ratio $|\mathcal{M}_{n'-1}|/|\mathcal{M}_{n'}|$ is always bounded above by a polynomial in n', hence Facts 2.1 and 2.2 imply efficient schemes for the Eulerian orientation problems.

However, in order to argue about the cardinalities of $\mathcal{M}_{n'}$ and $\mathcal{M}_{n'-1}$ that arise in the reduction, we need to introduce some further terminology.

- Let P denote an orientation, not necessarily Eulerian, of an Eulerian graph $G=(V,E)$. We may regard P as a directed graph $P=(V,F)$, where if $\{u,v\} \in E$ then either $(u,v) \in F$ (the edge is directed towards v), or $(v,u) \in F$ (the edge is directed out of v).

- For an orientation $P=(V,F)$ and a vertex $v \in V$, the *charge of v* is

$$q(v) = \frac{|\{u : (v,u) \in F\}| - |\{u : (u,v) \in F\}|}{2} \quad .$$

Clearly, $-\frac{d(v)}{2} \leq q(v) \leq \frac{d(v)}{2}$, where $d(v)$ is the degree of v in G. A vertex with zero charge is called *balanced.* For example, the charge of a positively charged vertex v is the number of edges directed out of v that must be reversed in order for v to become balanced.

- For an orientation $P=(V,F)$, let $V^+ = \{v \in V : q(v) > 0\}$, and $V^- = \{v \in V : q(v) < 0\}$. The *charge of P* is

$$\vartheta P = \sum_{v \in V^+} q(v) = \sum_{v \in V^-} q(v) \quad ,$$

where the last equality is easy to verify from first principles. By analogy to near-perfect matchings, we shall call orientations with charge one *near-Eulerian.*

- Finally, let

$$\mathcal{P}_k = \{P : \vartheta P = k\} \quad ,$$

where $\mathcal{P}_k = \emptyset$ for $k > |E|/2$. Furthermore, by flow preservation considerations, it is not hard to verify that for any $P \in \mathcal{P}_k$ there is at least one set of $2k$ edge-disjoint paths from vertices in V^+ to vertices in V^-.

In some sense, orientations in $\mathcal{P}_k$ are flows "more severely constrained" than orientations in $\mathcal{P}_{k-1}$, and one might expect this to reflect upon the cardinalities of $\mathcal{P}_k$ and $\mathcal{P}_{k-1}$. In Monotonicity Lemma 3.3 we formalize this behavior in full generality, and show that for any Eulerian graph G, and for any k, $|\mathcal{P}_k| \leq n(n-1)|\mathcal{P}_{k-1}|$. This last bound is polynomially related to the ratio $|\mathcal{M}_{n'-1}|/|\mathcal{M}_{n'}|$ for graphs that arise from the reduction of counting Eulerian orientations to counting perfect matchings (Lemmas 3.1 and 3.2), thus we obtain efficient sampling and approximate counting schemes for Eulerian orientations of any Eulerian graph.

We proceed to establish the approximation results.

3 Sampling and Approximate Counting

In this section we present efficient sampling and approximate counting schemes for the set $\mathcal{P}_0$ of Eulerian orientation of any Eulerian graph $G=(V,E)$, where $|V|=n$ and $|E|=m$. The reduction of this problem to a perfect matchings problem is as follows:

Let $G' = (V_1', V_2', E')$ be a bipartite graph with vertex bipartition $V_1' = \bigcup_{v\in V} X_v$, where $X_v = \cup_{e\in E: e=\{u,v\}}\{x_{v,e}\}$, and $V_2' = \bigcup_{e\in E}\{w_e\} \cup \bigcup_{v\in V} Y_v$, where $Y_v = \{y_{v,i}\,,\ 1 \le i \le \frac{d(v)}{2}\}$, and edges $E' = \bigcup_{e\in E: e=\{u,v\}} \{\{x_{u,e}, w_e\}, \{x_{v,e}, w_e\}\} \cup \bigcup_{v\in V} X_v \times Y_v$. Clearly $n' = |V_1'| = |V_2'| = \sum_{v\in V} d(v) = 2m$, hence the size of G' is polynomial in the size of G.

LEMMA 3.1. *For any Eulerian graph G, the set $\mathcal{M}_{n'}$ of perfect matchings of G' can be partitioned so that the partition classes are in one-to-one relation with the set $\mathcal{P}_0$ of Eulerian orientations of G. Furthermore each partition class has cardinality $\prod_{v\in V}\left(\frac{d(v)}{2}\right)!$. [Hence, we can efficiently approximate the number of Eulerian orientations of G if we can efficiently approximate the number of perfect matchings of G' and divide by $\prod_{v\in V}\left(\frac{d(v)}{2}\right)!$.]*

PROOF. Note first that each perfect matching of G' can be associated with a unique Eulerian orientation of G as follows: For each edge $\{u,v\}$ of G, there is a pair of edges $\{x_{u,e}, w_e\}$ and $\{x_{v,e}, w_e\}$ in G', exactly one of which must be in the perfect matching to cover w_e. If $\{x_{v,e}, w_e\}$ is in the perfect matching of G' then direct $\{u,v\}$ as (u,v) in the orientation of G, otherwise direct $\{u,v\}$ as (v,u). Thus a unique orientation of G is obtained. To see that this is indeed Eulerian, notice that for each v, all $\frac{d(v)}{2}$ vertices in Y_v must be matched to $\frac{d(v)}{2}$ of the $d(v)$ vertices in X_v, which forces the remaining $\frac{d(v)}{2}$ vertices in X_v that remain unmatched to be matched to their associated w_e's, which clearly balances the incoming and outgoing edges of the corresponding orientation. Hence perfect matchings can be partitioned according to the Eulerian orientation that they are associated with.

Furthermore, it is easy to see by similar reasoning that each Eulerian orientation gives rise to a nonempty partition class of perfect matchings, and in fact, that each partition class has cardinality exactly $\prod_{v\in V}\left(\frac{d(v)}{2}\right)!$, since once the w_e's are matched with half of the X_v's for each v, there are $\left(\frac{d(v)}{2}\right)!$ ways of matching the other half of the X_v's with the Y_v's. □

LEMMA 3.2. *For any Eulerian graph G, $|\mathcal{M}_{n'-1}|/|\mathcal{M}_{n'}| = \mathcal{O}\left(n^2|\mathcal{P}_1|/|\mathcal{P}_0|\right)$, where $\mathcal{P}_0$ and $\mathcal{P}_1$ are Eulerian and near-Eulerian orientations of G, and $\mathcal{M}_{n'}$ and $\mathcal{M}_{n'-1}$ are perfect and near-perfect matchings of G'. [Hence, by Lemma 3.1 and Facts 2.1 and 2.2 we can efficiently approximate the number of Eulerian orientations of G if $|\mathcal{P}_1|/|\mathcal{P}_0|$ can be upperbounded by a polynomial in n.]*

PROOF. By tedious but straightforward reasoning as in Lemma 3.1, it can be shown that $|\mathcal{M}_{n'-1}| \le 2m|\mathcal{M}_{n'}| + \left(\frac{d_{\max}}{2}+1\right)^2 \prod_{v\in V}\left(\frac{d(v)}{2}\right)!\,|\mathcal{P}_1|$, and the proof follows (details are left for the full paper). □

LEMMA 3.3. **(Monotonicity Lemma.)** *For any Eulerian graph $G=(V,E)$, where $|V|=n$, and for any k such that $1\le k\le |E|/2$,*

$$|\mathcal{P}_k| \le n(n-1)|\mathcal{P}_{k-1}| \ .$$

PROOF. Let q be a realizable "charge function" on V, i.e. suppose there exists at least one orientation such that each vertex $v\in V$ has charge $q(v)$. Let $\mathcal{P}_k(q)$ be the set of orientations such that each vertex v has charge $q(v)$. Clearly all orientations in $\mathcal{P}_k(q)$ have the same V^+ and V^-, and clearly $k=\sum_{v\in V^+} q(v)$. For the next few paragraphs we assume that q is fixed, and V^+ and V^- are with respect to the fixed q. Finally, let $\mathcal{Q}_q$ be the set of charge functions that can be obtained from q by decreasing $q(u)$ by one for some $u\in V^+$, increasing $q(v)$ by one for some $v\in V^-$, and leaving q otherwise the same. We shall show that the following holds for any fixed q:

$$|\mathcal{P}_k(q)| \le |\bigcup_{q'\in\mathcal{Q}_q} \mathcal{P}_{k-1}(q')| \qquad (3.1)$$

To prove (1) we shall relate orientations in $\mathcal{P}_k(q)$ to orientations in $\bigcup_{q'\in\mathcal{Q}_q}\mathcal{P}_{k-1}(q')$ so that, roughly, the number of orientations in $\bigcup_{q'\in\mathcal{Q}_q}\mathcal{P}_{k-1}(q')$ related with each orientation in $\mathcal{P}_k(q)$ is larger than the number of orientations in $\mathcal{P}_k(q)$ related with each orientation in $\bigcup_{q'\in\mathcal{Q}_q}\mathcal{P}_{k-1}(q')$. Of course, the natural way to relate orientations in $\mathcal{P}_k(q)$ with orientations in $\bigcup_{q'\in\mathcal{Q}_q}\mathcal{P}_{k-1}(q')$ is by reversing a path with endpoints $u\in V^+$ and $v\in V^-$. In what follows we introduce a technique to decompose an Eulerian orientation into so-called "Euler elements" so that the number of paths to be reversed can be easily accounted for. The decomposition in Euler elements is as follows:

(I). A *circuit* of an orientation $P=(V,F)$ is a sequence (w_1,w_2), (w_2,w_3), …, (w_{l-1},w_l), (w_l,w_1), where all the (w_i, w_{i+1})'s are distinct edges in F. [Note that, e.g. (w_1, w_2), (w_2, w_1), (w_1, w_3), (w_3, w_1) is the same circuit as (w_2, w_1), (w_1, w_3), (w_3, w_1), (w_1, w_2), i.e., a circuit is a generalized cycle: there is no starting point.]

(Ia). A circuit of P is a *free circuit* if no edge is incident to a vertex in $V^+ \cup V^-$.

(Ib). A circuit of P is a V^+*-circuit* if there are exactly two edges in the circuit incident to some vertex in V^+: one incoming and one outgoing, and no edge is incident to a vertex in V^-. Analogously, define V^-*-circuits.*

(II). A (V^+, V^-)*-path* of P is a sequence (u, w_1), (w_1, w_2), (w_2, w_3), …, (w_{l-1}, w_l), (w_l, v), where (u, w_1), (w_m, v), and all the (w_i, w_{i+1})'s are distinct edges in F; moreover, $u \in V^+$, $v \in V^-$, and all the w_i's (which, as in circuits, are not necessarily distinct) are not in $V^+ \cup V^-$. [Note that, in contrast to circuits, a (V^+, V^-)-path has a specified starting point: u, and an ending point: v).] Finally, (V^-, V^+)*-paths,* (V^+, V^+)*-paths, and* (V^-, V^-)*-paths* are analogously defined.

(III). An *Euler element* of P is either a free circuit, or a V^+-circuit, or a V^--circuit, or a (V^+, V^-)-path, or a (V^-, V^+)-path, or a (V^+, V^+)-path, or a (V^-, V^-)-path.

We argue that each orientation $P=(V, F) \in \mathcal{P}_k(q) \cup \bigcup_{q' \in \mathcal{Q}_q} \mathcal{P}_{k-1}(q')$ has exactly $\prod_{w \in V \setminus V^+ \cup V^-} \left(\frac{d(w)}{2}\right)!$ distinct partitionings of its edges into Euler elements. To see this, for each vertex $w \in V \setminus V^+ \cup V^-$ consider one of the $\left(\frac{d(w)}{2}\right)!$ pairings of the incoming and outgoing edges in w. For some vertex w, let φ_w denote such a pairing. The partition of F into Euler elements that is suggested by some fixed $\varphi = \{\varphi_w : w \in V \setminus V^+ \cup V^-\}$ is as follows:

- Free circuits are of the form: $(w_1, w_2), (w_2, w_3) = \varphi_{w_2}(w_1, w_2), \ldots, (w_l, w_1) = \varphi_{w_l}(w_{l-1}, w_l), (w_1, w_2) = \varphi_{w_1}(w_l, w_1)(w_1, w_2)$, where the w_i's $\in V \setminus V^+ \cup V^-$.
- V^+-circuits are of the form: $(u, w_1), (w_1, w_2) = \varphi_{w_1}(u_1, w_1), \ldots, (w_l, u) = \varphi_{w_l}(w_{l-1}, w_l)$, where $u \in V^+$ and the w_i's $\in V \setminus V^+ \cup V^-$. And, of course, V^--circuits are of analogous form.
- (V^+, V^-)-paths are of the form: $(u, w_1), (w_1, w_2) = \varphi_{w_1}(u, w_1), \ldots, (w_l, v) = \varphi_{w_l}(w_{l-1}, w_l)$, where the $u \in V^+$, $v \in V^-$, and the w_i's $\in V \setminus V^+ \cup V^-$. And, clearly, (V^-, V^+), (V^+, V^+), and (V^-, V^-)-paths are of analogous form.

It is not hard to see that this is indeed a partition of F into Euler elements, and that two distinct sets of pairings φ suggest different partitions. Furthermore, there are $\prod_{w \in V \setminus V^+ \cup V^-} \left(\frac{d(w)}{2}\right)!$ distinct sets of pairings φ for each P in $\mathcal{P}_k(q) \cup \bigcup_{q' \in \mathcal{Q}_q} \mathcal{P}_{k-1}(q')$.

Now to prove (1) it suffices to show

$$\left(\prod_{w \in V \setminus V^+ \cup V^-} \left(\frac{d(w)}{2}\right)!\right) |\mathcal{P}_k(q)| \leq \left(\prod_{w \in V \setminus V^+ \cup V^-} \left(\frac{d(w)}{2}\right)!\right) |\bigcup_{q' \in \mathcal{Q}_q} \mathcal{P}_{k-1}(q')|$$

Consider $\prod_{w \in V \setminus V^+ \cup V^-} \left(\frac{d(w)}{2}\right)!$ copies of each orientation in $\mathcal{P}_k(q) \cup \bigcup_{q' \in \mathcal{Q}_q} \mathcal{P}_{k-1}(q')$, one for each way φ of pairing and partitioning the edges of the orientation. Say that $\langle P, \varphi \rangle$, where $P \in \mathcal{P}_k(q)$, is related to $\langle P', \varphi' \rangle$, where $P' \in \bigcup_{q' \in \mathcal{Q}_q} \mathcal{P}_{k-1}(q')$, if and only if P can be obtained from P' by reversing one of the (V^+, V^-)-paths in P' that are suggested by φ', moreover $\varphi' = \varphi$. It is easy to see that if there are $2k+c$ (V^+, V^-)-paths in P' that are suggested by φ', then there must also be c (V^-, V^+)-paths, and that if $\langle P, \varphi \rangle$ and $\langle P', \varphi' \rangle$ are related they have identical decompositions suggested by $\varphi = \varphi'$, except for the one reversed path that related them which is a (V^+, V^-)-path in P' and a (V^-, V^+)-path in P. Hence if $\langle P', \varphi' \rangle$ is related to $2k+c$ $\langle P, \varphi \rangle$'s, then each one of these $\langle P, \varphi \rangle$'s is related to at most $c+1$ $\langle P', \varphi' \rangle$'s. Now it is easy to see that (2) holds true, because $k > 1$ and hence $2k+c \geq c+1$, and by elementary counting considerations, and hence (1) also holds true.

The following final step where we allow q to vary completes the proof of Lemma 3.3:

$$\begin{aligned} |\mathcal{P}_k| &= \sum_{q'} |\mathcal{P}_k(q')| \\ &\leq n(n-1) \sum_{q} \sum_{q' \in \mathcal{Q}_q} |\mathcal{P}_k(q')| \\ &\leq n(n-1) \sum_{q} |\mathcal{P}_{k-1}(q)| \\ &= n(n-1)|\mathcal{P}_{k-1}| \end{aligned}$$

□

THEOREM 3.1. *For any Eulerian graph $G=(V, E)$, where $|V| = n$, there are (d, δ)-sampling and (ϵ, δ)-approximate counting schemes for the set $\mathcal{P}_0$ of Eulerian orientations of G. The running time is polynomial in n, ϵ^{-1}, $\log d^{-1}$, and $\log \delta^{-1}$.*

PROOF. Follows by Lemmas 3.1, 3.2, an 3.3, and Facts 2.1 and 2.2. □

REMARK. We prove Lemma 3.3 for general k, while in the proof of Theorem 3.4 only the special case $k = 1$ is needed. In the following section we need both cases

$k = 1$ and $k = 2$, and in a potential general scheme that estimates $|\mathcal{P}_0|$ directly rather than through the reduction to perfect matchings Lemma 3.3 is needed in full generality. We shall have more to say about this in Remark 3 at the end of Section 4.

4 Euler Decompositions and further Path Arguments

Theorem 3.4 suggests a sampling scheme for the set $\mathcal{P}_0$ by simulating a sampling scheme for $\mathcal{M}_{n'}$. As proposed in [3], the latter sampling scheme is a Markov chain on $\mathcal{M}_{n'} \cup \mathcal{M}_{n'-1}$ which possesses the so-called rapid mixing property (this property has by now received much attention; the unfamiliar reader is referred [1,3,10,16,17] etc. for details). In this section we outline how to obtain a direct sampling scheme for $\mathcal{P}_0$, by simulating a rapidly mixing Markov chain $\mathcal{MC}$ on $\mathcal{P}_0 \cup \mathcal{P}_1$. Except for the definition of $\mathcal{MC}$ (which is rather obvious) the new technical point here is the use of decompositions into Euler elements for the definition of "random canonical paths", so that the expected path congestion can be bounded by the "path encoding" method of Jerrum and Sinclair [10]. In particular,

Let $\mathcal{MC}$ be a Markov chain on state space $\mathcal{P}_0 \cup \mathcal{P}_1$ and transition probabilities defined as follows:

```
Let P(t)=(V,F(t)) be the state on time t;
Toss a fair coin;
If "heads" then P(t+1):=P(t) else
    Pick (u,v)∈F(t) uniformly at random;
    If P'=(V,F(t)\{(u,v)}∪{(v,u)})∈P_0∪P_1
            then P(t+1):=P'
            else P(t+1):=P(t);
```

So the action of $\mathcal{MC}$ is to move unit charges along circuits and paths, thus walking from one orientation to another by reversing the edges that the unit charges have traversed.

It is trivial to check that $\mathcal{MC}$ is symmetric, irreducible, and aperiodic, therefore it converges to the uniform distribution over $\mathcal{P}_0 \cup \mathcal{P}_1$. It is significantly more involved to show that $\mathcal{MC}$ has the rapid mixing property:

THEOREM 4.1. *For any Eulerian graph G and any time t:*

$$d(t) = \sum_{P \in \mathcal{P}_0 \cup \mathcal{P}_1} |\Pr[P(t)=P] - \frac{1}{|\mathcal{P}_0 \cup \mathcal{P}_1|}| \le 2^{\mathcal{O}(m)} e^{-\Omega(\frac{t}{m^2 n^6})} .$$

PROOF. (Outline). We shall first define paths beween any pair of Eulerian and near Eulerian orientations and we shall bound path congestion through any particular edge. This, in turn, implies rapid mixing following standard arguments [4,5,6,10].

Let $P_A = (V, F_A) \in \mathcal{P}_0$, and let $P_B = (V, F_B) \in \mathcal{P}_1$ with $q(u) = +1$ and $q(v) = -1$. Consider the "difference" $P_{AB} : P_{AB} = (V, F_A \oplus F_B \setminus F_A)$ (where $\oplus$ denotes symmetric difference). Notice that in P_{AB} all vertices in $V \setminus \{u, v\}$ have even degrees and they are balanced (same number of incoming and outgoing edges); while u must have odd degree with one more outgoing than incoming edge, and similarly v must have one more incoming than outgoing edge. For a balanced vertex w in P_{AB}, let $d_{AB}(w)$ denote the in-degree=out-degree of w, and let $\varphi_{AB}(w)$ be a pairing of incoming with outgoing edges in w. There are clearly $d_{AB}(w)!$ distinct $\varphi_{AB}(w)$'s, and $\prod_{w \in V \setminus \{u,v\}} d_{AB}(w)!$ distinct φ_{AB}'s, where $\varphi_{AB} = \{\varphi_{AB}(w) : w \in V \setminus \{u, v\}\}$. Following the terms and the reasoning of Lemma 3.3, each φ_{AB} induces a decomposition of P_{AB} in $c+1$ $(\{u\}, \{v\})$-paths, c $(\{v\}, \{u\})$-paths, and c' circuits which include free circuits, $\{u\}$-circuits, and $\{v\}$-circuits.

Define a "φ_{AB}-canonical" path from P_A to P_B as one that first reverses the circuits (in any order); then reverses the paths by alternating reverses of $(\{u\}, \{v\})$-paths with reverses of $(\{v\}, \{u\})$-paths, and starting from a $(\{u\}, \{V\})$-PATH (and with no other restriction on the order). A circuit is reversed by reversing its edges successively one at a time, starting from a (u, u') edge for $\{u\}$-circuits, a (v, v') edge for $\{v\}$-circuits, and the lexicographically first edge for a free circuit. A $(\{u\}, \{v\})$-path is reversed by reversing its edges successively one at a time starting from u. A $(\{v\}, \{u\})$-path is reversed by reversing its edges successively one at a time starting from v. So a φ_{AB}-canonical path from P_A to P_B consists indeed of a sequence of steps on states of $\mathcal{MC}$. There are $c'!(c + 1)!c!$ distinct φ_{AB}-canonical paths from P_A to P_B.

For each pair $P_A \in \mathcal{P}_0$ and $P_B \in \mathcal{P}_1$, pick a "representative" path from P_A to P_B by first choosing some φ_{AB} uniformly at random, and then chosing a φ_{AB}-canonical path uniformly at random.

Let $(P_M, P_{M'})$ be an edge in $\mathcal{MC}$. Let $E(P_M, P_{M'})$ denote the expected number of pairs (P_A, P_B) such that the representative path from P_A to P_B uses $(P_M, P_{M'})$. It can be argued along the lines of [4,5] that

$$E(P_M, P_{M'}) \le |\mathcal{P}_0 \cup \mathcal{P}_1| \mathcal{O}(n^4) ,$$

and the bound on $d(t)$ of the theorem follows by the above, combined with a bound in [6]:

$$d(t) \le |\mathcal{P}_0 \cup \mathcal{P}| \left(1 - \frac{|\mathcal{P}_0|}{|\mathcal{P}_1| dlc}\right)^t ,$$

where d^{-1} is the transition probability which in our case is m^{-1}, l is the length of the canonical paths which in our case is m, and c is $E(P_M, P_{M'})$. □

REMARK 1. A sampling scheme for $\mathcal{P}_0$ can be obtained by simulating $\mathcal{MC}$ for t_0 steps, so that $d(t_0)$ is sufficiently small, and if $P(t_0) \in \mathcal{P}_0$ output $P(t_0)$, otherwise claim fail and repeat. Roughly, one out of n^2 simulations yields $P(t_0) \in \mathcal{P}_0$, and the whole scheme is efficient.

REMARK 2. The sampling scheme for $\mathcal{P}_0$ through the reduction to perfect matchings is obtained by simulating Broder's Markov chain on $\mathcal{M}_{n'} \cup \mathcal{M}_{n'-1}$ [3] for the graph G' of Section 3. The convergence rate of this chain is $2^{\mathcal{O}(m)} e^{-\Omega(\frac{t}{m^2 n^8})}$, and, since $|\mathcal{M}_{n'-1}|/|\mathcal{M}_{n'}| = \mathcal{O}(n'^4)$, only one out of n^4 simulations yields a perfect matching. Hence simulating $\mathcal{MC}$ for sampling $\mathcal{P}_0$ is faster than the reduction to perfect matchings by a factor of n^4.

REMARK 3. Markov chains analogous to $\mathcal{MC}$ can be defined for $\mathcal{Q}_k = \cup_{i=0}^{k} \mathcal{P}_k$ for all $1 \leq k \leq |E|/2$. If all these chains are rapidly mixing, then a direct scheme to approximate $\mathcal{P}_0$ can be obtained by **(i)** simulating these chains to obtain estimates $r_1 \simeq |\mathcal{P}_0|/|\mathcal{Q}_1|$, $r_2 \simeq |\mathcal{Q}_1|/|\mathcal{Q}_2|$, $\ldots$, $r_{|E|/2} \simeq |\mathcal{Q}_{|E|/2-1}|/|\mathcal{Q}_{|E|/2}|$, (since by Lemma 3.3 all the r_i's are $\Omega(n^2)$) and, **(ii)** since $|\mathcal{Q}_{|E|/2}| = 2^m$, use $|\mathcal{P}_0| \simeq 2^m \prod_i r_i$. However, establishing the rapid mixing property for all these chains remains open.

5 Hardness of Exact Counting

THEOREM 5.1. *Exact counting for Eulerian orientations is #P-complete.*

PROOF. We give a reduction from counting perfect matchings. Let $G = (V_1, V_2, E)$ be a bipartite graph, $|V_1| = |V_2| = n$, $|E| = m$, and let $\mathcal{M}_n$ be the set of perfect matchings of G. It has been shown in [18] that computing $|\mathcal{M}_n|$ is #P-complete. In what follows we argue that computing $|\mathcal{M}_n|$ can be achieved in polynomial time with $\lfloor m - 2n \rfloor + 1$ calls to an oracle that counts Eulerian orientations.

Assume without loss of generality that each vertex of G has degree at least two.

Consider a graph G' on vertex set $V_1 \cup V_2 \cup \{s, t\}$, and edges $E \cup E_s \cup E_t$, where E_s consists of edges connecting s to V_1, so that for each edge $v \in V_1$ there are $d(v) - 2$ parallel edges connecting s to v in G', and similarly for E_t connect each $v \in V_2$ to t with $d(v) - 2$ parallel edges. Hence there are $\sum_{v \in V_1} (d(v) - 2) = \sum_{v \in V_2} (d(v) - 2) = m - 2n = m'$ edges connecting s to V_1 and t to V_2. Finally consider the graph $G_{m'}$ which is G' together with m' parallel edges connecting s to t, and in general G_k is G' together with k parallel edges connecting s to t.

Now realize that in an Eulerian orientation of $G_{m'}$ such that all parallel edges between s and t are directed from t to s, all edges connecting s to V_1 must be oriented out of s, while all edges connecting V_2 to t must be oriented towards t. Consequently each vertex $v \in V_1$ which receives $d(v) - 2$ incoming edges from s must receive exactly one incoming edge from V_2, while the rest $d(v) - 1$ of the edges connecting v to V_2 must be oriented out of v. Similarly, for each vertex $u \in V_2$ there is exactly one edge connecting u to V_1 that is oriented out of u. So clearly, the edges between V_1 and V_2 that are oriented out of V_2 and into V_1 define a perfect matching of the original graph G, and clearly each perfect matching of G gives rise to a unique Eulerian orientation in $G_{m'}$ where all edges connecting s to V_1 are oriented out of s, while all edges connecting V_2 to t are oriented towards t.

Let $\mathcal{X}_j$ denote the set of orientations of G' (not necessarily Eulerian) such that all vertices in V_1 and V_2 are balanced, and exactly j of the m' edges connecting s to V_1 are oriented out of s, while j of the m' edges connecting t to V_2 are oriented towards t. We have established that

$$(5.1) \qquad |\mathcal{X}_{m'}| = |\mathcal{M}_n| \ .$$

Furthermore symmetry implies that

$$(5.2) \qquad |\mathcal{X}_k| = |\mathcal{X}_{m'-k}| \qquad m' \geq k > \lfloor \frac{m'}{2} \rfloor \ .$$

Now if $\mathcal{P}_0(G_k)$ denotes the set of Eulerian orientations of G_k, where $k = 0, 2, \ldots, m'$ if m' is even, and $k = 1, 3 \ldots, m'$ if m' is odd, then the cardinalities of the $\mathcal{P}_0(G_k)$'s can be written in terms of the cardinalities of the $\mathcal{X}_j$'s as follows:

$$(5.3) \qquad |\mathcal{P}_0(G_k)| = \sum_{i=0}^{k} \binom{k}{i} |\mathcal{X}_{\frac{m'+k}{2} - i}| \ ,$$

where (5) holds since there are $\binom{k}{i}$ ways of directing i parallel edges between s and t from s to t and the remaining $k - i$ from t to s, and this forces exactly $\frac{m'+k}{2} - i$ edges between s and V_1 to be directed from s to V_1 (so that s is balanced).

Finally the $|\mathcal{P}_0(G_k)|$'s can be inferred by $\lfloor m - 2n \rfloor + 1$ calls to an oracle that counts Eulerian orientations (one query for each graph G_k), hence (3), (4), and (5) yield $M' + 2$ independent equations in unknowns $|\mathcal{X}_k$, $k = 0, \ldots, m'$, and $|\mathcal{M}_n|$, from which $|\mathcal{M}_n|$ can be deduced (it is easy to see that the system can be solved in polynomial time) . □

Finally we provide the following connection between the number of Eulerian orientations and the number of Euler tours: Let $G = (V, E)$ be an Eulerian (undirected) graph. Let φ_v be a partition of the edges incident to v into pairs (notice that G is undirected). Let $\varphi = \{\varphi_v : v \in V\}$. Realize that φ imposes a decomposition/partition of the edges of G where a partition class consists of $\{e=\{u,v\}, \varphi_u(e), \varphi_v(e), \ldots\}$. Let $\mathcal{T}_k = |\{\varphi : \varphi \text{ induces } k \text{ partition classes }\}|$. Notice that if φ induces one partition class, then this suggests exactly 2 Euler tours for G, so $\mathcal{T}_1 = (\#\text{Euler tours})/2$. It is not hard to verify that

$$(5.4) \qquad \prod_{v \in V} \left(\frac{d(v)}{2}\right)! \, |\mathcal{P}_0| = \sum_{k=1}^{|E|/2} 2^k \mathcal{T}_k \quad .$$

Hence, by Theorem 5.1, some of the $\mathcal{T}_k$'s must be hard to compute. It might me worth investigating (6) further to specify which $\mathcal{T}_k$'s are provably hard.

6 Summary

We gave efficient randomized schemes to sample and approximately count Eulerian orientations, as well as the complementary hardness result. The following questions are interesting to investigate further: **(i)** Extend the results for graphs with capacities on the edges; here we treat the case with capacities 1. **(ii)** Classify the complexity of counting problems for Euler tours. **(iii)** To what extend are our results parallelizable? **(iv)** Resolve Remark 3 in Section 4.

7 Acknowledgements

Special thanks are due to Dominic Welsh for some very enlightening discussions, and to Madhu Sudan for directing us to Welsh's work. Thanks are also due to Madhu Sudan, Prasad Tetali, Umesh Vazirani, and Vijay Vazirani for many discussions during the early stages of this work.

References

[1] D. Aldous, On the Markov chain Simulation Method for Uniform Combinatorial Distributions and Simulated Annealing, *Probability in Eng. and Inf. Sci.* 1, 1987, pp 33-46.

[2] R. Baxter, Exactly Solved Methods in Statistical Mechanics, *Academic Press*, London.

[3] A.Z. Broder, How Hard is it to Marry at Random? (On the Approximation of the Permanent), *STOC* 1986, pp 50-58.

[4] P. Dagum and M. Luby, Approximating the Permanent of Graphs with Large Factors, *Siam J. on Comp*, to appear.

[5] P. Dagum, M. Luby, M. Mihail, and U. Vazirani, Polytopes, Permanents, and Graphs with Large Factors, *FOCS* 1988, pp 412-421.

[6] P. Diaconis and D.Strock, Geometric Bounds for Eigenvalues of Matkov Chains, preprint.

[7] M. Dyer, A. Frieze, and R. Kannan, A Random Polynomial Time Algorithm for Estimating Volumes of Convex Bodies,*STOC* 1989, pp 375-381.

[8] F. Jaeger, D.L. Vertigan, and D.J.A. Welsh, On the Computational Complexity of Jones and Tutte Polynomials, *Math. Proc. Camb. Phil. Soc.* (1990), 108, 35, pp 35-53.

[9] M.R. Jerrum, 2 Dimentional Monomer-Dimer Systems are Computationally Intractable, *Journal of Statistical Physics* 48, pp 121-134.

[10] M.R. Jerrum and A. Sinclair, Conductance and the Rapid Mixing property for Markov chains: the Approximation of the Permanent resolved, *STOC* 1988, pp 235-243.

[11] M.R. Jerrum and A. Sinclair, Fast Uniform Generation of Regular Graphs, *ICALP Proceedings* 1989.

[12] M.R. Jerrum and A. Sinclair, Polynomial-time Approximation Algorithms for the Ising Model, Tech. Report, University of Edinburgh, (1990).

[13] M.R. Jerrum, L.G. Valiant, and V.V. Vazirani, Random Generation of Combinatorial Structures from a Uniform Distribution, *Theoretical Computer Science* 43, 1986, pp 169-188.

[14] E.H. Lieb, Residual Entropy of Square Ice, *Physical Review* 162, 162-171.

[15] L. Lovász and M. Simonovis, The Mixing Rate of Markov Chains, An Isoperimetric Inequality, and Computing the Volume, *FOCS* 1990, to appear.

[16] M. Mihail, Conductance and Convergence of Markov Chains (A Combinatorial Treatment of Expanders), *FOCS* 1989, pp 526-531.

[17] A. Sinclair and M.R. Jerrum, Approximate counting, uniform generation and rapidly mixing markov chains, *Information and Computing*, to appear.

[18] L.G. Valiant, The Complexity of Computing the Permanent, *Theoretical Computer Science*, 1979, pp 189-201.

[19] D.J.A. Welsh, "On the Tutte Map", Lecture Notes, DIMACS May '91 and Dagstuhl June '91.

[20] D.J.A. Welsh, The Computational Complexity of Some Classical Problems from Statistical Physics, in *Disorder in Physical Systems*, Oxford University Press, (1990), pp 307-321.