Abstract—Mechanism design for distributed systems is fundamentally concerned with aligning individual incentives with social welfare to avoid socially inefficient outcomes that can arise from agents acting autonomously. One simple and natural approach is to centrally broadcast non-binding advice intended to guide the system to a socially near-optimal state while still harnessing the incentives of individual agents. The analytical challenge is proving fast convergence to near optimal states, and in this paper we give the first results showing that carefully constructed advice vectors could yield stronger guarantees.

We apply this approach to a broad family of potential games modeling vertex cover and set cover optimization problems in a distributed setting. This class of problems is interesting because finding exact solutions to their optimization problems is NP-hard yet highly inefficient equilibria exist, so a solution in which agents simply locally optimize is not satisfactory. We show that with an arbitrary advice vector, a set cover game quickly converges to an equilibrium with cost of the same order as the square of the social cost of the advice vector. More interestingly, we show how to efficiently construct an advice vector with a particular structure with cost $O(\log n)$ times the optimal social cost, and we prove that the system quickly converges to an equilibrium with social cost of this same order.

I. INTRODUCTION

Mechanism design for distributed systems is fundamentally concerned with aligning individual incentives with social welfare to avoid socially inefficient outcomes that can arise from agents acting autonomously. One simple and natural approach is to centrally broadcast non-binding advice intended to guide the system to a socially near-optimal state while still harnessing the incentives of individual agents.

This paper focuses on a natural set cover game. As a concrete example, say a state’s legislature wants to establish a number of subsidized health clinics. Residents in a county that houses such a clinic will enjoy its benefits, but they will also incur additional local taxes to pay for the clinic. Residents in a county without a clinic do not incur additional taxes, but they only receive the benefits of a clinic if there is one in a neighboring county. We would like a particular set of clinics to open in each county so as to optimize the aggregate cost-benefit calculation for the state. However, since clinics are locally subsidized, counties individually decide whether to open a clinic, so we cannot centrally dictate a near-optimal distribution of clinics. This paper shows how to advertise an overall strategy (determined centrally and using global knowledge) so that even if self-interested counties are only influenced by their advertised strategies with a small probability and for a short time period, they will eventually reach a socially near-optimal solution in a distributed way.

Set covering problems are important and interesting from a classical optimization point of view, but also as a game theoretic framework both for analyzing social problems like the one described above, where agents behave autonomously in some natural self-interested way, and for engineering distributed systems in which locally-aware agents can be programmed to behave in this way. In this paper, we model covering problems as games, and we use models from learning theory to describe local decision making by players in these games. We are interested in demonstrating convergence not to arbitrary local equilibria but to states that are low cost relative to the global optimum. We accomplish this by incorporating a globally-informed central authority into natural behavior dynamics.

A. Game Setup, Equilibrium Quality, and Dynamic Models

Given a universe of elements with associated costs and a collection of sets of these elements, the minimum weighted set cover optimization problem is to choose the lowest cost subset of elements such that each set is represented by at least one chosen element. While this problem is NP-hard, good approximation algorithms exist; however, such algorithms do not take into account individual incentives.

We analyze a setting in which a central authority knows a subset of elements that approximates an optimal solution to the set cover problem, but elements are modeled as only locally aware agents with cost functions representing a natural distributed game interpretation of the core optimization problem. We generalize the problem by not requiring total coverage, rather the importance of covering a given set is determined by its set weight. Each element $i$ that chooses to be on incurs his own cost $c_i$, and each element $i$ that is off pays the sum of the weights of sets he participates in that do not contain any other on element. If the element costs are all smaller than the set weights, then the cost-minimizing set of on elements is also the optimal set cover. If additionally each set is of size two, then this is the special case of a minimum weighted vertex cover problem.

The healthcare example above illustrates a social network in which agents have inherent costs associated with being on (pay for a local clinic) or off (hope your neighbor pays for a clinic), which are correlated with the social objective. Another motivation consists of engineering networks
in which non-willful distributed agents are programmed to make decisions based on their surroundings. The extensive literature on cooperative control has shown that in this setting many optimization problems can be conveniently solved in a distributed fashion by endowing agents with artificial individual objective functions and cost-minimizing behavior [6]. As a concrete example, our set cover games include non-cooperative power management models in wireless sensor networks [3]. The elements are autonomous sensors, and a geographic region is a set consisting of elements corresponding to sensors that could cover that region. A sensor that is on is charged some fixed cost, whereas a sensor that is off is charged a cost proportional to the number or importance of its adjacent regions that are uncovered by any other sensor.

Much of the work on cooperative control and dynamics-based algorithmic game theory only guarantees that systems converge to an arbitrary equilibrium. Many games, however, have a high Price of Anarchy (PoA), where PoA is the ratio of the social costs of the worst equilibrium and the globally optimal configuration. The following special case illustrates that PoA is $\Omega(n)$ in our set cover game:

**Example 1 (Star graph):** Suppose $n$ agents (or players) are charged some amount $c < 1$ when they are on and otherwise penalized 1 for every incident uncovered set. Then a star graph in which vertices are agents and edges are sets has a global optimum with only the center on, yielding social cost $c$, compared to a low quality Nash Equilibrium in which only the center is off, yielding social cost $c(n-1)$.

To circumvent such a high PoA, behavioral models incorporating advertising effects have been recently proposed [1], [2]. The models share the common feature that a central authority has knowledge of some joint strategy profile with low social cost, and this authority broadcasts this strategy in the hopes that players will adopt their prescribed strategies. Specifically, the public service advertising model (PSA) of [1] assumes that each agent independently has an $\alpha$ probability of receiving and temporarily adopting the advertising strategy. Those that do not receive and adopt their prescribed strategy behave in a myopic best response manner. This model is well-suited for an engineering systems setting, where we do not expect all components to receive the central authority’s signal. The learning models of [2] assume that each agent uses any of a broad class of learning algorithms to continually choose between acting according to their local best response move and their broadcasted signal. In the learn-then-decide (LTD) model, agents eventually commit to one of these options. In contrast to PSA, LTD is better motivated by a social setting where agents that are only locally aware are interested in exploring the advertising strategy with the hopes that it will benefit them personally. In this work, we study both PSA and LTD models.

**B. Our Contribution**

We show indeed that the high price of anarchy results for the case of vertex cover and set cover games can be largely circumvented. Furthermore, we give the first theoretical results for PSA and LTD models that leverage particular structural aspects of the advice vector $s^{ad}$. Specifically, we show the following.

**R1:** For the vertex cover games and arbitrary advice $s^{ad}$, agents converge to a state of expected cost $O(\text{cost}(s^{ad}))$ in the PSA and LTD models.

**R2:** For set cover games and arbitrary advice $s^{ad}$, agents converge to a state of expected cost $O(\text{cost}(s^{ad})^2)$ in the PSA and LTD models (with different constants).

**R3:** For set cover games and and some carefully chosen advice $s^{ad}$, agents converge to a state of cost $O(\text{cost}(s^{ad}))$ with high probability in PSA model. Furthermore, we provide a poly-time algorithm to construct such an advice vector $s^{ad}$ that is an $O(\log n)$ approximation to the optimal social cost configuration.

We emphasize that all the above convergence guarantees happen in a polynomial number of steps in terms of the number of agents.

**II. Preliminaries**

**A. Background on General Games**

We represent a general game as a triple $G = \langle N, (S_i), (\text{cost}_i) \rangle$, where $N$ is a set of $n$ players, $S_i$ is the finite action space of player $i \in N$, and $\text{cost}_i$ denotes the cost function of player $i$. The joint action space of the players is $S = S_1 \times \cdots \times S_n$. For a joint action $s \in S$, we denote by $s_{-i}$ the actions of all players $j \neq i$. Players’ cost functions map joint actions to non-negative real numbers, i.e. $\text{cost}_i : S \rightarrow \mathbb{R}^+$ for all $i \in N$. In this paper, we define a social cost function, $\text{cost} : S \rightarrow \mathbb{R}$, simply as the summation of individual players’ costs. The optimal social cost is denoted by

$$OPT = \min_{s \in S} \text{cost}(s).$$

Given a joint action $s$, the best response of player $i$ is the set of actions that minimizes player $i$’s cost subject to the other players’ fixed actions $s_{-i}$, i.e.

$$BR_i(s_{-i}) = \arg\min_{s_i \in S_i} \text{cost}_i(s_i, s_{-i}).$$

**Best response dynamics** is a process in which at each time step, an arbitrary player not already playing best response updates his action to one in his current best response set. A joint action $s \in S$ is a pure Nash equilibrium if no player $i \in N$ can benefit from deviating to another action, namely, $s_i \in BR_i(s_{-i})$ for every $i \in N$.

A game $G$ is called an exact potential game [4] if there exists a potential function $\Phi : S \rightarrow \mathbb{R}$ such that

$$\text{cost}_i(a', s_{-i}) - \text{cost}_i(a, s_{-i}) = \Phi(a', s_{-i}) - \Phi(a, s_{-i}),$$

for all $i \in N$, $s_{-i} \in S_{-i}$, and $a, a' \in S_i$. For general potential games, only the signs of both sides of these equations must be equal. While general games are not guaranteed to have a pure Nash equilibrium, all finite potential games do and furthermore best response dynamics in such

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1As mentioned earlier, a set cover game where each set has size $2$ is called a vertex cover game, and in such games equilibria have natural connections to vertex covers in the graph induced by the sets (i.e. edges).
games always converges to a pure Nash equilibrium [4], [5]. However, the convergence time can be exponentially large in terms of the number of players in general.

Two well known concepts for quantifying the inefficiency of equilibria relative to non-equilibria are Price of Anarchy and Price of Stability. For \( \mathcal{N}(G) \) the set of pure Nash equilibria of game \( G \), Price of Anarchy (PoA) and Price of Stability (PoS) are defined as:

\[
\text{PoA} = \max_{s \in \mathcal{N}(G)} \frac{\text{cost}(s)}{\text{OPT}} \quad \text{PoS} = \min_{s \in \Lambda(\sigma)} \frac{\text{cost}(s)}{\text{OPT}}.
\]

B. Covering Game

A cover game \( G = ([n], (S_i), (\text{cost}_i)) \) is specified by a collection of sets \( F \subseteq 2^{[n]} \), costs \( c_i \) for \( i \in [n] \), and weights \( w_\sigma \) for \( \sigma \in F \). For all \( i \in [n] \), we have \( S_i = \{\text{on}, \text{off}\} \), and we define \( \text{cost}_i \) in equation (1) after introducing some additional notation for this game.

We say a set \( \sigma \in F \) is covered in joint strategy \( s \in S \) if \( s_i = \text{on} \) for some \( i \in \sigma \). Otherwise, \( \sigma \) is said to be uncovered. Denote the collection of sets that include agent \( i \) and are uncovered in \( s \) with \( F^u_i(s) \), dropping the \( s \) when clear from context. The entire set of uncovered sets is written \( F^u := \bigcup_i F^u_i \). For \( \sigma \subseteq [n] \), define \( c(\sigma) := \sum_{i \in \sigma} c_i \), and for \( F' \subseteq F \), define \( w(F') := \sum_{\sigma \in F'} w_\sigma \). We make use of the following shorthand: \( c_{\max} := \max_i c_i \), \( c_{\min} := \min_i c_i \), \( w_{\max} := \max_{\sigma \in F} w_\sigma \), \( w_{\min} := \min_{\sigma \in F} w_\sigma \), and \( F_{\max} = \max_{\sigma \in F} |\sigma| \), and we always assume these quantities are \( \Theta(1) \) unless otherwise noted. Now we define the cost function of agent \( i \) with respect to any joint strategy \( s \in S \): \( \text{cost}_i(s) = \begin{cases} c_i & \text{if } s_i = \text{on} \\ w(F^u_i) & \text{if } s_i = \text{off}. \end{cases} \)

For a joint action (or strategy profile) \( s \in S \), let ON\( (s) \) and OFF\( (s) \) be sets of nodes that are \( \text{on} \) and \( \text{off} \), respectively. Then the social cost has the following simple form:

\[
\text{cost}(s) = \sum_{i \in [n]} \text{cost}_i(s) = c(\text{ON}(s)) + \sum_{\sigma \in F^u} |\sigma| \cdot w_\sigma.
\]

Observe that \( c_i \) expresses how costly it is for agent \( i \) to cover the sets that contain him. For example, if \( c_{\max} < w_{\min} \), it will always be cheaper for an agent to be \( \text{on} \) than to participate in an uncovered sets and every set will be covered in equilibrium. The socially optimal equilibrium will necessarily be the minimum cost set cover\(^2\). Finally, we let \( \Delta_k = \Delta_k(G) \) denote the \( k \)-th order maximum degree of the hypergraph induced by \( F \):

\[
\Delta_k := \max_{i_1, \ldots, i_k \in [n]} |\{\sigma \in F : \{i_1, \ldots, i_k\} \subseteq \sigma, i_j \text{ distinct}\}|.
\]

\(^2\)We note that if we simply redefine the costs so that \( i \) pays \( c_i \) if he is \( \text{off} \) and pays the sum of the weights of the fully-covered sets he participates in if he is \( \text{on} \), this game is a packing analog of the original cover game. The equilibria when \( c_{\max} < w_{\min} \) are configurations in which no set is fully covered; in particular, when \( F_{\max} = 2 \), the sets of \( \text{on} \) agents in any equilibrium is a maximal independent set.

Recall that best response dynamics converge to pure Nash equilibria for potential games. Now observe that the cover game is an exact potential game with potential function

\[
\Phi(s) = c(\text{ON}(s)) + w(F^u(s)).
\]

Combining this with the social cost formula, we have that for any \( s \in S \)

\[
\Phi(s) \leq \text{cost}(s) \leq F_{\max} \cdot \Phi(s).
\]

C. Optimization and Equilibrium Quality

The star graph example from the introduction reveals that PoA in the cover game can be \( \Omega(n) \). This motivates the need for efficient dynamics with better guarantees than convergence to arbitrary equilibria.

As a step in that direction, we propose a centralized, poly-time LP-rounding algorithm to find a low-cost configuration \( s^{ad} \) for the cover game:

1. Compute \( x^* \) as

\[
\arg\min_{x \in [0,1]^n} \sum_{i \in [n]} c_i \cdot x_i \text{ s.t. } \sum_{i \in \sigma} x_i \geq 1 \forall \sigma \in F \]

2. Set \( s^ad_i = \begin{cases} \text{on} & \text{if } x^*_i \geq 1/F_{\max} \\ \text{off} & \text{otherwise} \end{cases} \)

Lemma 1: The configuration \( s^{ad} \) obtained from LP-rounding has \( \text{cost}(s^{ad}) \leq F_{\max} \cdot \frac{c_{\max}}{w_{\min}} \cdot \text{OPT} \).

Proof: Let \( s^* \) be some joint strategy that achieves optimal social cost, and let \( s' \) be the joint strategy obtained by turning \( \text{on} \) an arbitrary element in each set \( \sigma \) that is uncovered in \( s^* \). Then we have \( \text{cost}(s^{ad}) \leq F_{\max} \cdot \sum_i c_i \cdot x_i^* \leq F_{\max} \cdot \text{cost}(s') \leq F_{\max} \cdot \frac{c_{\max}}{w_{\min}} \cdot \text{cost}(s^*) = F_{\max} \cdot \frac{c_{\max}}{w_{\min}} \cdot \text{OPT} \).

Assuming constant weights and costs, this is an \( O(F_{\max}) \)-approximation.

III. Public Service Advertising

In this section and the following one, we show that price of anarchy is avoidable in cover games even using best response-inspired dynamics as long as these dynamics incorporate some form of suggestion from a weak central authority that is aware of a high quality equilibrium.

The first model we study in this paper is the public service advertising (PSA) model in [1] in which a central authority broadcasts a strategy for each agent, which some agents receive and temporarily follow. Player behavior is described in two phases:

1. Play begins in an arbitrary state, and a central authority advertises joint action \( s^{ad} \in S \). Each agent receives the proposed strategy independently with probability \( \alpha \in (0,1) \). Agents that receive this signal are called receptive. They play their advertising strategies throughout Phase 1, and non-receptive agents undergo best response dynamics to settle on a joint strategy that is a Nash equilibrium given the fixed behavior of receptive agents. We call this joint strategy \( s' \).

2. All agents participate in best response dynamics until convergence to some Nash equilibrium \( s'' \).
Since our cover game is a potential game and all potential games eventually converge to a Nash equilibrium under best response dynamics, both phases are guaranteed to terminate. Furthermore, convergence occurs in poly-time with respect to parameters \( \{n, \{c_i\}, \{w_i\}\} \).

A. Effect of Advertising in PSA

In this section we show that advertising helps significantly in cover games. In particular, we show that if the advertising strategy \( s^{ad} \) has low social cost, then the cost of the resulting equilibrium is low even if only a small constant \( \alpha \) fraction of the agents receive and respond to the signal. Theorem 2 formalizes the general result of this section, and Theorem 4 improves this result for particular advertising strategies. We first present the formal results of the section and then prove these two theorems. Note that the proof of Theorem 2 introduces notation used throughout. Recall that we assume for convenience that costs and weights are all \( \Theta(1) \).

Theorem 2: For any PSA advertising strategy \( s^{ad} \),

\[
E[\text{cost}(s'')] \leq \begin{cases} O(\Delta_2) \cdot \text{cost}(s^{ad})^2 & \text{if } F_{\max} = O(1) \\ O(1) \cdot \text{cost}(s^{ad}) & \text{if } F_{\max} = 2. \end{cases}
\]

For \( s^{ad} \) obtained from the \( O(F_{\max}) \)-approximation polytime LP-rounding algorithm of Section II-B, the following corollary is immediate.

Corollary 3: There exists a poly-time algorithm to find an advertising strategy \( s^{ad} \) for the PSA model such that

\[
E[\text{cost}(s'')] \leq \begin{cases} O(\Delta_2) \cdot \text{OPT}^2 & \text{if } F_{\max} = O(1) \\ O(1) \cdot \text{OPT} & \text{if } F_{\max} = 2. \end{cases}
\]

We additionally consider advertising strategies particular to our game for improved performance of the model. For advertising strategy \( s^{ad} \), define the core minimum degree of \( \text{on} \) elements \( \Delta_1^* := \Delta_1^*(s^{ad}) \) to be the minimum over all \( \text{on} \) elements in \( s^{ad} \) of the number of sets in which such element is the unique \( \text{on} \) element. We say strategy \( s^{ad} \) satisfies Condition (\( \star \)) if for all \( x \geq \frac{\Delta_1^*}{\Delta_2(F_{\max}-1)} \), we have

\[
\left( \frac{C_{\text{max}}}{w_{\text{min}}} + 1 \right) x \left[ \frac{\text{max}}{\text{min}} \right] + 1 \left( 1 - \alpha F_{\max} \right) x - \left[ \frac{\text{max}}{\text{min}} \right] - 1 \leq \frac{1}{n^2}, \quad (\star)
\]

Configurations satisfying this condition are efficient in the sense that each \( \text{on} \) element is individually responsible for covering many sets.

We can construct a low-cost advertising strategy satisfying Condition (\( \star \)) by first observing that any joint strategy \( s \) with \( \Delta_1(s) \geq B\Delta_2\log n \) for a large enough constant \( B \) (depending on constants \( C_{\text{max}}/w_{\text{min}}, \alpha, F_{\max} \)) satisfies Condition (\( \star \)). Then let \( s' \) be the strategy with social cost \( O(1) \cdot \text{OPT} \) obtained by LP-rounding, and construct \( s^{ad} \) satisfying Condition (\( \star \)) with social cost \( O(\Delta_2\log n) \cdot \text{OPT} \) by greedily turning off every agent that is the unique \( \text{on} \) element in fewer than \( B\Delta_2\log n \) sets in \( s' \).

3This is because \( \Phi \) is bounded above and below by functions of these parameters and decreases under best response dynamics.

Theorem 4 is a stronger, high probability version of Theorem 2 when \( s^{ad} \) satisfies Condition (\( \star \)), and the previous procedure for constructing such \( s^{ad} \) makes Corollary 5 immediate.

Theorem 4: For any PSA advertising strategy \( s^{ad} \) satisfying Condition (\( \star \)), if \( F_{\max} = O(1) \), then with probability at least \( 1 - 1/n \)

\[
\text{cost}(s'') = O(1) \cdot \text{cost}(s^{ad}).
\]

Corollary 5: There exists a poly-time algorithm to find an advertising strategy \( s^{ad} \) for the PSA model such that if \( F_{\max} = O(1) \), then with probability at least \( 1 - 1/n \)

\[
\text{cost}(s'') = O(\Delta_2\log n) \cdot \text{OPT}.
\]

B. Proof of Theorem 2

We begin with some notation. We say two agents contained in a common set are neighbors. \( L \) and \( R \) denote the sets of agents that are \( \text{on} \) and \( \text{off} \), respectively, in \( s^{ad} \). \( L_{\text{off}} \) and \( R_{\text{on}} \) denote the set of agents in \( L \) that are \( \text{off} \) in \( s' \) and the set of agents in \( R \) that are \( \text{on} \) in \( s' \), respectively. \( F_R \) denotes the collection of sets uncovered in \( s^{ad} \) and \( F_{\text{bad}} \) denotes the collection of sets not in \( F_R \) but uncovered in \( s' \).

From (5) and \( F_{\max} = O(1) \), any sequence of best response moves increases social cost by at most a constant factor. All agents best respond in Phase 2, and hence \( \text{cost}(s'') = O(\text{cost}(s')) \). It suffices to bound \( \text{cost}(s') \). At a high level, we do this by bounding the expectation of \( \omega(F_{\text{bad}}) \), the total weight of sets covered in \( s^{ad} \) but uncovered in \( s' \) (Lemma 6), and \( |R_{\text{on}}| \), the number of agents that are \( \text{off} \) in \( s^{ad} \) but \( \text{on} \) at \( s' \) (Lemma 7).

From (2), the constant cost and weight assumption, and \( F_{\max} = O(1) \), we have

\[
E[\text{cost}(s')] \leq \text{cost}(s^{ad}) + E[c(R_{\text{on}})] + F_{\max} \cdot E[\omega(F_{\text{bad}})]
\]

\[
= \text{cost}(s^{ad}) + O \left( \frac{E[c(R_{\text{on}})]}{|R_{\text{on}}|} \right) + O \left( E[\omega(F_{\text{bad}})] \right).
\]

This with the following two lemmas bound \( \text{cost}(s') \) relative to \( \text{cost}(s^{ad}) \), completing the proof of Theorem 2.

Lemma 6: \( \omega(F_{\text{bad}}) \leq c(L) \).

Proof: Each set in \( F_{\text{bad}} \) should contain an element in \( L \) that is best responding and \( \text{off} \) in \( s' \). Hence, \( \omega(F_{\text{bad}}) = \sum_{\ell \in \text{Log}} \sum_{u \in F_u} \omega(u) \leq \sum_{\ell \in \text{Log}} c_{\ell} \leq c(L) \), completing the proof of Lemma 6.

Lemma 7: \( n \cdot |R_{\text{on}}| \leq F_R + O(2|L|) \).

Proof: We consider three types of sets at the end of Phase 1 such that each \( r \in R_{\text{on}} \) must be in one of these types of sets, and we bound the number of each type of set. The last of these bounds is given in expectation over the set of agents receptive to advertising.

Since each element \( R_{\text{on}} \) plays best response in \( s' \), we can associate each \( r \in R_{\text{on}} \) with a distinct set \( \sigma_r \) in which \( r \) is the only \( \text{on} \) element. Define

\[
R_{\text{on}}^{(1)} = \{ \sigma_r : \sigma_r \subseteq R \},
\]
and observe that by definition, 
\[ |R^{(1)}_{\text{on}}| \leq |F_R|. \]  
(7)

Every other set \( \sigma_r \notin R^{(1)}_{\text{on}} \) has at least one element in \( L \) and all such elements are \( \text{off} \). Each of these sets is one of the following types:

\[ R^{(2)}_{\text{on}} = \{ \sigma_r \in F : \sigma_r \cap L \subseteq L_{\text{off}}, |\sigma_r \cap L| > 1 \} \]
\[ R^{(3)}_{\text{on}} = \{ \sigma_r \in F : \sigma_r \cap L \subseteq L_{\text{off}}, |\sigma_r \cap L| = 1 \} \]

Note that \( \sigma \in R^{(2)}_{\text{on}} \) contains at least three elements, so there are no such sets when \( F_{\text{max}} = 2 \). Recall that in general, each pair of agents is in at most \( \Delta_2 \) common sets, so since there are at most \( |L|^2 \) pairs of agents in \( L \), we have

\[ |R^{(2)}_{\text{on}}| \leq \Delta_2 \cdot |L|^2 \text{ if } F_{\text{max}} = O(1), \]
\[ 0 \text{ if } F_{\text{max}} = 2. \]  
(8)

Let \( F^*_\ell \) denote the collection of sets containing \( \ell \) as the unique element in \( L \). Since each set in \( R^{(3)}_{\text{on}} \) contains a unique element \( \ell \in L \) and this element is \( \text{off} \) in \( s' \), it follows that

\[ E[|R^{(3)}_{\text{on}}|] = \sum_{\ell \in L} |F^*_\ell| \cdot \Pr[\ell \text{ is off}]. \]  
(9)

Further, we observe that

\[ \Pr[\ell \text{ is off}] \leq \Pr[|\{ \rho \in F^*_\ell : \rho \cap R \text{ is off} \}| \leq c_{\text{max}}/w_{\text{min}}] \]
\[ \leq \Pr[|\{ \rho \in F^*_\ell : \rho \cap R \text{ is receptive} \}| \leq c_{\text{max}}/w_{\text{min}}] \]
\[ \leq \Pr[|\{ \rho \in F^*_\ell : \rho \cap \text{off} \text{ is receptive} \}| \leq c_{\text{max}}/w_{\text{min}}], \]

where \( F^*_\ell \) is a subset of \( F^*_\ell \) in which each \( r \in R \) is present at most one set in \( F^*_\ell \). From definitions of \( F_{\text{max}} \) and \( \Delta_2 \), we can choose this collection of sets so that \( |F^*_\ell| \leq (F_{\text{max}} - 1)\Delta_2 \cdot |F^*_\ell| \). Importantly, since no pair of sets in \( F^*_\ell \) have common elements in \( R \), the probability that any \( \rho \in F^*_\ell \) has \( \rho \cap R \text{ all receptive} \) is independent of the probability that any other \( \rho' \in F^*_\ell \) has \( \rho' \cap R \text{ all receptive} \), so we can further bound

\[ \Pr[\ell \text{ is off}] \leq \Pr[|\{ \rho \in F^*_\ell : \rho \cap R \text{ is receptive} \}| \leq c_{\text{max}}/w_{\text{min}}] \]
\[ \leq \sum_{i=0}^{\max} \binom{i}{F^*_\ell} \left( 1 - \alpha F_{\text{max}} \right)^i F^*_\ell^{-i} \left( \alpha F_{\text{max}} \right)^i. \]  
(10)

Combining (9) and (10), we get

\[ E[|R^{(3)}_{\text{on}}|] \leq \sum_{\ell \in L} |F^*_\ell| \cdot \sum_{i=0}^{\max} \binom{i}{F^*_\ell} \left( 1 - \alpha F_{\text{max}} \right)^i F^*_\ell^{-i} \left( \alpha F_{\text{max}} \right)^i \]
\[ \leq (F_{\text{max}} - 1)\Delta_2 \]
\[ \times \sum_{\ell \in L} \sum_{i=0}^{\max} |F^*_\ell| \left( 1 - \alpha F_{\text{max}} \right)^i F^*_\ell^{-i} \left( \alpha F_{\text{max}} \right)^i \]
\[ = O(\Delta_2) \cdot |L|, \]  
(11)

where the last equality is from the following proposition, proven in the appendix.

**Proposition 8:** For constant \( a \in (0, 1) \) and \( 0 < c \leq d, \)

\[ \sum_{i=0}^{\max} \binom{d}{i} (1 - a)^d - i \alpha^i = O(\binom{d}{i}). \]

Finally, since \( |R_{\text{on}}| \leq |R^{(1)}_{\text{on}}| + |R^{(2)}_{\text{on}}| + |R^{(3)}_{\text{on}}| \) by construction, (7), (8) and (11) together give the desired conclusion of Lemma 7 noting that \( \Delta_2 = 1 \) when \( F_{\text{max}} = 2 \).

**C. Extension to Unbounded Costs and Weights**

The assumption of constant weights and costs is merely for convenience of exposition, and all the results and proof techniques in this paper naturally extend more general to weights and costs. In particular, one can obtain the following theorem (analogous to Theorem 2) in the PSA model by accounting for costs and weights explicitly in each step in the proof of Theorem 2.

**Theorem 9:** For any advertised strategy \( s^{ad} \) in PSA,

\[ E[\text{cost}(s')] \leq \begin{cases} O\left( \Delta_2 \frac{c_{\text{max}}}{w_{\text{min}}} \right) \text{cost}(s^{ad})^2 & \text{if } F_{\text{max}} = O(1) \\ O\left( \frac{c_{\text{max}}}{w_{\text{min}}} \right) \text{cost}(s^{ad}) & \text{if } F_{\text{max}} = 2 \end{cases} \]

**D. Proof of Theorem 7**

In proving our stronger result for advertising strategies satisfying Condition (\( \ast \)), we use the same notation presented in the beginning of the proof of Theorem 2. It again suffices to bound the cost of \( s' \). In particular, we will prove that \( \text{cost}(s') = O(\text{cost}(s^{ad})) \) with all but at most \( 1/n \) probability.

Recall that \( \text{cost}(s^{ad}) \geq c(L) + w(F_R) \). The following lemma proves that for \( s^{ad} \) satisfying Condition (\( \ast \)), all agents in \( L \) turn on with probability at least \( 1 - 1/n \), the cost of agents in \( R_{\text{on}} \) is bounded by \( w(F_R) \leq \text{cost}(s^{ad}) \) in this event, establishing the desired conclusion of Theorem 4.

**Lemma 10:** If the advertising strategy \( s^{ad} \) satisfies Condition (\( \ast \)), then \( F_{\text{bad}} = \emptyset \) and \( c(R_{\text{on}}) \leq w(F_R) \) with probability at least \( 1 - 1/n \).

**Proof:** As in the proof Lemma 7 (and using the same notation), for any \( \ell \in L \) there is some subset \( F^*_\ell \subseteq F^*_\ell \) such that no pair of sets in \( F^*_\ell \) have common elements in \( R \) and \( |F^*_\ell| \geq \frac{|F^*_\ell|}{(F_{\text{max}} - 1)\Delta_2} \). Then as we derived in in the proof Lemma 7

\[ \Pr[\ell \text{ is off}] \leq \sum_{i=0}^{\max} \binom{i}{F^*_\ell} \left( 1 - \alpha F_{\text{max}} \right)^i F^*_\ell^{-i} \left( \alpha F_{\text{max}} \right)^i, \]

and by assumption that \( s^{ad} \) satisfies (\( \ast \)), the above expression is at most \( 1/n^2 \). By union bound, \( \Pr[L_{\text{off}} = 0] \geq 1 - 1/n \) and hence \( F_{\text{bad}} = \emptyset \).

Now assume the event that all nodes in \( L \) are \( \text{on} \). Observe that for each best responding \( r \in R_{\text{on}}, c_r \) is no greater than the total weight of all sets containing \( r \) as the unique \( \text{on} \) agent. Since we assume all nodes in \( L \) are \( \text{on} \) these sets are a subset of \( F_R \). Further, since there is no overlap in these
sets between different agents in \( R_{on} \), we can sum over all \( r \in R_{on} \) to derive \( c(R_{on}) \leq w(T_r) \). This completes the proof of Lemma 10.

IV. LEARN-THEN-DECIDE

We also study the set cover game in the learn-then-decide (LTD) model of [2]. In contrast to PSA, agents in LTD are neither strictly receptive nor strictly best responders in the initial exploration phase, but they choose one of these options for the final exploitation phase:

1: Play begins in an arbitrary state, and a central authority advertises joint action \( s^{ad} \in S \). Player \( i \) is associated with fixed probability \( p_i \geq \beta \in (0,1) \). Agents are chosen to update uniformly at random for each of \( T^* \) time steps. When \( i \) updates, he plays \( s^{ad}_i \) with probability \( p_i \) or best response with probability \( 1 - p_i \). The state at time \( T^* \) is denoted \( s' \).

2: At time \( T^* \), all agents in random order individually commit arbitrarily to \( s^{ad}_i \) or best response. Then agents take turns in random order playing their chosen strategy until they reach a Nash equilibrium \( s'' \) given the fixed behavior of \( s^{ad} \) followers.

A. Effect of Advertising in LTD

The first phase of the LTD model lends itself to similar yet somewhat more complicated techniques used to analyze the first phase of the PSA model. However, since the second phase is not merely best response dynamics, we must develop new techniques to analyze that cost stays low in this phase, and here we lose an additional \( \Delta_2 \) factor. Due to space restrictions, we omit the proof of Theorem 11 which can be found in the full version. Corollary 12 is immediate from the LP-approximation algorithm.

**Theorem 11:** There exists a \( T^* \in \text{poly}(n) \) such that for any advertising strategy \( s^{ad} \) in the LTD model,

\[
E[\text{cost}(s'')] \leq \begin{cases} O(\Delta_3^2) \cdot \text{cost}(s^{ad})^2 & \text{if } F_{\text{max}} = O(1) \\ O(1) \cdot \text{cost}(s^{ad}) & \text{if } F_{\text{max}} = 2 \end{cases}.
\]

**Corollary 12:** There exists a poly-time algorithm to find an advertising strategy \( s^{ad} \) for the LTD model such that

\[
E[\text{cost}(s'')] \leq \begin{cases} O(\Delta_3^2) \cdot \text{OPT}^2 & \text{if } F_{\text{max}} = O(1) \\ O(1) \cdot \text{OPT} & \text{if } F_{\text{max}} = 2 \end{cases}.
\]

V. CONCLUSIONS

In recent years, game theoretic frameworks have provided informative models for analyzing the outcomes of games among autonomous agents or components programmed as autonomous agents. However, many games, including those studied in this paper, often suffer from high Price of Anarchy, meaning that without a central authority it is hard to induce a state with low social cost. In this paper we study how weak broadcasting signals from a central authority are enough to induce states with low social cost in a general class of covering problems. Our techniques could be of broader interest for analyzing other classic optimization problems in a distributed fashion.

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REFERENCES


APPENDIX

A. Proof of Proposition 8

Proposition 8 is immediate \( c < 1 \) since \( d(1 - a)^d = O(1) \) for all \( d \geq 0 \) as long as \( a \in (0,1) \) is constant. Hence, assume \( c \geq 1 \). Let \( \tilde{a} = \max(a, 1 - a) \) and define \( \xi \in (0,1) \) to be the largest constant satisfying

\[
(\xi^2) \xi < \sqrt{\frac{1}{\tilde{a}}}
\]

For each \( \ell \), we have either \( d \leq c/\xi \) or \( d > c/\xi \). For the case with \( d \leq c/\xi \), observe that with \( c \leq d \), the desired expression is at most

\[
d \sum_{i=0}^{d} \binom{d}{i} (1 - a)^{d-i} a^i = d = O(c)
\]

Now consider when \( d > c/\xi \). Observe that

\[
d \sum_{i=0}^{c} \binom{d}{i} (1 - a)^{d-i} a^i \leq d \cdot \tilde{a}^d \sum_{i=0}^{c} \binom{d}{i} \leq d \cdot \tilde{a}^d \sum_{i=0}^{c} \frac{d^i}{i!}
\]

Further, we have

\[
d \cdot \tilde{a}^d \sum_{i=0}^{c} \frac{d^i}{i!} = O(1) \cdot \tilde{a}^d/2 \sum_{i=0}^{c} \frac{d^i}{i!}
\]

where we use (a) \( d \cdot \tilde{a}^d/2 \) is \( O(1) \), (b) \( d^i/i! \) is increasing with respect to \( i \) for \( i < c < d \), (c) \( x! = \Omega((x/e)^x) \), (d) \( c < \xi \cdot d \) and (e) the definition of \( \xi \). This completes the proof of Proposition 8.