1 Bayesian Linear Regression Prediction (50pts)

In the Bayesian linear regression problem, we assume that our data points \((x_1, y_1), \ldots, (x_n, y_n)\) are generated as:

\[ y_t = \theta^T x_t + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2), \]

with a prior on parameter \(\theta\) given by \(N(\mu_\theta, \Sigma_\theta)\).

Given an \(x_t\), we would like to predict \(y_t\). In class, we showed, using linearity of expectation, that the mean of the predictive distribution over \(y_t\) was given by:

\[
\mathbb{E}[y_t] = \mathbb{E}[\theta^T x_t + \epsilon] = \mathbb{E}[\theta^T x_t] + \mathbb{E}[\epsilon] = \mathbb{E}[\theta^T x_t] + 0 = \mu_\theta^T x_t
\]

What is the variance of \(y_t\)?

Hint: For uncorrelated random variables \(X_i\)

\[
\text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i)
\]
2  Conditional Independence in a Gaussian

In the Natural / Canonical Parameterization of a Gaussian, $P$'s sparsity pattern is essentially encoding the graphical model structure of the Gaussian random variables.

2.1  (30pts)
Show that if $P_{ij} = 0$, then conditioned on everything else, $x_i$ and $x_j$ are independent.

2.2  (20pts)
Argue by example why $P_{ij} = 0$ does not imply independence even if it implies conditional independence.