Fundamental matrix

Let $p$ be a point in left image, $p'$ in right image.

Epipolar relation

- $p$ maps to epipolar line $l'$
- $p'$ maps to epipolar line $l$

Epipolar mapping described by a 3x3 matrix $F$

\[ p' F p = 0 \]
Fundamental matrix

This matrix \( F \) is called

- the “Essential Matrix”
  - when image intrinsic parameters are known
- the “Fundamental Matrix”
  - more generally (uncalibrated case)

Can solve for \( F \) from point correspondences

- Each \((p, p')\) pair gives one linear equation in entries of \( F \)

\[
p'Fp = 0
\]

- \( F \) has 9 entries, but really only 7 or 8 degrees of freedom.
- With 8 points it is simple to solve for \( F \), but it is also possible with 7. See [Marc Pollefeys’s notes](#) for a nice tutorial
The scale of algorithm name quality

better

RANSAC
SIFT
Deep Learning
Optical Flow
Hough Transform
Neural Networks
Essential and Fundamental Matrix

worse

Dynamic Programming
Today’s lecture

• Stereo Matching (Spare correspondence to Dense Correspondence)
• Optical Flow (Dense motion estimation)
Stereo Matching
Stereo image rectification
Stereo image rectification

- Reproject image planes onto a common plane parallel to the line between camera centers.
- Pixel motion is horizontal after this transformation.
- Two homographies (3x3 transform), one for each input image reprojection.

Rectification example
The correspondence problem

• Epipolar geometry constrains our search, but we still have a difficult correspondence problem.
Fundamental Matrix + Sparse correspondence

Photo Tourism
Exploring photo collections in 3D

Noah Snavely  Steven M. Seitz  Richard Szeliski
University of Washington  Microsoft Research

SIGGRAPH 2006
The Visual Turing Test for Scene Reconstruction
Supplementary Video

Qi Shan+    Riley Adams+    Brian Curless+
Yasutaka Furukawa*    Steve Seitz++

+University of Washington    *Google

3DV 2013
Despite their scale invariance and robustness to appearance changes, SIFT features are local and do not contain any global information about the image or about the location of other features in the image. Thus feature matching based on SIFT features is still prone to errors. However, since we assume that we are dealing with rigid scenes, there are strong geometric constraints on the locations of the matching features and these constraints can be used to clean up the matches. In particular, when a rigid scene is imaged by two pinhole cameras, there exists a $3 \times 3$ matrix $F$, the Fundamental matrix, such that corresponding points $x_{ij}$ and $x_{ik}$ (represented in homogeneous coordinates) in two images $j$ and $k$ satisfy $^{10}$:

$$x_{ij}^T F x_{ij} = 0.$$  (3)

A common way to impose this constraint is to use a greedy randomized algorithm to generate suitably chosen random estimates of $F$ and choose the one that has the largest support among the matches, i.e., the one for which the most matches satisfy (3). This algorithm is called Random Sample Consensus (RANSAC)$^6$ and is used in many computer vision problems.
### Sparse to Dense Correspondence

<table>
<thead>
<tr>
<th>Input images</th>
<th>SfM points</th>
<th>MVS points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Colosseum</td>
<td></td>
<td></td>
</tr>
<tr>
<td>St. Peter's</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Building Rome in a Day**

By Sameer Agarwal, Yasutaka Furukawa, Noah Snavely, Ian Simon, Brian Curless, Steven M. Seitz, Richard Szeliski

Communications of the ACM, Vol. 54 No. 10, Pages 105-112
Structure from motion (or SLAM)

- Given a set of corresponding points in two or more images, compute the camera parameters and the 3D point coordinates.
Structure from motion ambiguity

• If we scale the entire scene by some factor $k$ and, at the same time, scale the camera matrices by the factor of $1/k$, the projections of the scene points in the image remain exactly the same:

$$ x = PX = \left( \frac{1}{k} P \right)(kX) $$

It is impossible to recover the absolute scale of the scene!
How do we know the scale of image content?
Bundle adjustment

- Non-linear method for refining structure and motion
- Minimizing reprojection error

\[ E(P, X) = \sum_{i=1}^{m} \sum_{j=1}^{n} D(x_{ij}, P_iX_j)^2 \]
Correspondence problem

Multiple match hypotheses satisfy epipolar constraint, but which is correct?

Figure from Gee & Cipolla 1999
Correspondence problem

- Beyond the hard constraint of epipolar geometry, there are “soft” constraints to help identify corresponding points
  - Similarity
  - Uniqueness
  - Ordering
  - Disparity gradient

- To find matches in the image pair, we will assume
  - Most scene points visible from both views
  - Image regions for the matches are similar in appearance
Dense correspondence search

For each epipolar line
For each pixel / window in the left image
  • compare with every pixel / window on same epipolar line in right image
  • pick position with minimum match cost (e.g., SSD, normalized correlation)

Adapted from Li Zhang
Correspondence search with similarity constraint

- Slide a window along the right scanline and compare contents of that window with the reference window in the left image
- Matching cost: SSD or normalized correlation
Correspondence search with similarity constraint

Left

Right

scanline

SSD
Correspondence search with similarity constraint
Correspondence problem

- Clear correspondence between intensities, but also noise and ambiguity

Source: Andrew Zisserman
Correspondence problem

Neighborhoods of corresponding points are similar in intensity patterns.

Source: Andrew Zisserman
Correlation-based window matching
Correlation-based window matching

left image band \((x)\)
right image band \((x')\)
Correlation-based window matching

left image band \( x \)

right image band \( x' \)

cross correlation

disparity = \( x' - x \)
Correlation-based window matching

target region

left image band \((x)\)

right image band \((x')\)
Correlation-based window matching

Textureless regions are non-distinct; high ambiguity for matches.
Effect of window size

Source: Andrew Zisserman
Effect of window size

Want window large enough to have sufficient intensity variation, yet small enough to contain only pixels with about the same disparity.

Figures from Li Zhang
Results with window search

Window-based matching
(best window size)

Ground truth
Better solutions

- Beyond individual correspondences to estimate disparities:
- Optimize correspondence assignments jointly
  - Scanline at a time (DP)
  - Full 2D grid (graph cuts)
• What defines a good stereo correspondence?
  1. **Match quality**
     • Want each pixel to find a good match in the other image
  2. **Smoothness**
     • If two pixels are adjacent, they should (usually) move about the same amount
Stereo matching as energy minimization

\[ E = \alpha E_{\text{data}}(I_1, I_2, D) + \beta E_{\text{smooth}}(D) \]

\[ E_{\text{data}} = \sum_i (W_1(i) - W_2(i + D(i)))^2 \]

\[ E_{\text{smooth}} = \sum_{\text{neighbors } i,j} \rho(D(i) - D(j)) \]

- Energy functions of this form can be minimized using graph cuts


Source: Steve Seitz
Better results…

Graph cut method


For the latest and greatest: http://www.middlebury.edu/stereo/
Challenges

• Low-contrast; textureless image regions
• Occlusions
• Violations of brightness constancy (e.g., specular reflections)
• Really large baselines (foreshortening and appearance change)
• Camera calibration errors
Active stereo with structured light

- Project “structured” light patterns onto the object
  - Simplifies the correspondence problem
  - Allows us to use only one camera

Kinect: Structured infrared light

iPhone X
3 minute break
Variations on the Hermann grid: an extinction illusion

Jacques Ninio
Laboratoire de Physique Statistique\(^{(1)}\), École Normale Supérieure, 24 rue Lhomond, 75231 Paris cedex 05, France; e-mail: jacques.ninio@lps.ens.fr

Kent A Stevens
Department of Computer Science, Deschutes Hall, University of Oregon, Eugene, OR 97403, USA; e-mail: kent@cs.uoregon.edu
Received 21 September 1999, in revised form 21 June 2000

Abstract. When the white disks in a scintillating grid are reduced in size, and outlined in black, they tend to disappear. One sees only a few of them at a time, in clusters which move erratically on the page. Where they are not seen, the grey alleys seem to be continuous, generating grey crossings that are not actually present. Some black sparkling can be seen at those crossings where no disk is seen. The illusion also works in reverse contrast.

The Hermann grid (Brewster 1844; Hermann 1870) is a robust illusion. It is classically presented as a two-dimensional array of black squares, separated by rectilinear alleys. It is thought to be caused by processes of local brightness computation in arrays of
Computer Vision

Motion and Optical Flow

Many slides adapted from S. Seitz, R. Szeliski, M. Pollefeys, K. Grauman and others...
Video

- A video is a sequence of frames captured over time
- Now our image data is a function of space \((x, y)\) and time \((t)\)
Motion and perceptual organization

Gestalt psychology
(Max Wertheimer, 1880-1943)
Motion and perceptual organization

- Sometimes, motion is the only cue

Gestalt psychology (Max Wertheimer, 1880-1943)
Motion and perceptual organization

• Sometimes, motion is the only cue
Motion and perceptual organization

- Sometimes, motion is the only cue
Motion and perceptual organization

- Sometimes, motion is the only cue
Motion and perceptual organization

- Even “impoverished” motion data can evoke a strong percept
Motion and perceptual organization

- Even “impoverished” motion data can evoke a strong percept
Motion and perceptual organization


Courtesy of: Department of Psychology, University of Kansas, Lawrence.

Experimental study of apparent behavior. Fritz Heider & Marianne Simmel. 1944
Optic flow is the apparent motion of objects or surfaces.

Will start by estimating motion of each pixel separately.
Then will consider motion of entire image.
Problem definition: optical flow

How to estimate pixel motion from image $I(x,y,t)$ to $I(x,y,t+1)$?

- Solve pixel correspondence problem
  - given a pixel in $I(x,y,t)$, look for nearby pixels of the same color in $I(x,y,t+1)$

Key assumptions

- **color constancy**: a point in $I(x,y,t)$ looks the same in $I(x,y,t+1)$
  - For grayscale images, this is brightness constancy
- **small motion**: points do not move very far

This is called the optical flow problem
Optical flow constraints (grayscale images)

- Let's look at these constraints more closely
  - brightness constancy constraint (equation)
    \[ I(x, y, t) = I(x + u, y + v, t + 1) \]
  - small motion: \((u \text{ and } v \text{ are less than 1 pixel, or smooth})\)

Taylor series expansion of \(I\):

\[
I(x + u, y + v) = I(x, y) + \frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \text{[higher order terms]}
\]

\[
\approx I(x, y) + \frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v
\]
Optical flow equation

- Combining these two equations

\[ 0 = I(x + u, y + v, t + 1) - I(x, y, t) \]

\[ \approx I(x, y, t + 1) + I_x u + I_y v - I(x, y, t) \]

(Short hand: \( I_x = \frac{\partial I}{\partial x} \) for \( t \) or \( t+1 \))
Optical flow equation

- Combining these two equations

\[
0 = I(x+u, y+v, t+1) - I(x, y, t)
\]

\[
\approx I(x, y, t+1) + I_x u + I_y v - I(x, y, t)
\]

\[
\approx [I(x, y, t+1) - I(x, y, t)] + I_x u + I_y v
\]

\[
\approx I_t + I_x u + I_y v
\]

\[
\approx I_t + \nabla I \cdot <u, v>
\]

(Short hand: \(I_x = \frac{\partial I}{\partial x}\) for \(t\) or \(t+1\))
Optical flow equation

• Combining these two equations

\[ 0 = I(x+u, y+v, t+1) - I(x, y, t) \]

\[ \approx I(x, y, t+1) + I_x u + I_y v - I(x, y, t) \]

\[ \approx [I(x, y, t+1) - I(x, y, t)] + I_x u + I_y v \]

\[ \approx I_t + I_x u + I_y v \]

\[ \approx I_t + \nabla I \cdot <u, v> \]

In the limit as u and v go to zero, this becomes exact

\[ 0 = I_t + \nabla I \cdot <u, v> \]

(Brightness constancy constraint equation)

\[ I_x u + I_y v + I_t = 0 \]
How does this make sense?

**Brightness constancy constraint equation**

\[ I_x u + I_y v + I_t = 0 \]

- What do the static image gradients have to do with motion estimation?
The brightness constancy constraint

Can we use this equation to recover image motion \((u,v)\) at each pixel?

\[
0 = I_t + \nabla I \cdot <u,v> \quad \text{or} \quad I_x u + I_y v + I_t = 0
\]

- How many equations and unknowns per pixel?
  - One equation (this is a scalar equation!), two unknowns \((u,v)\)

The component of the motion perpendicular to the gradient (i.e., parallel to the edge) cannot be measured

If \((u, v)\) satisfies the equation, so does \((u+u', v+v')\) if

\[
\nabla I \cdot [u' \ v']^T = 0
\]
Aperture problem
Aperture problem
Aperture problem
The barber pole illusion

http://en.wikipedia.org/wiki/Barberpole_illusion
The barber pole illusion

http://en.wikipedia.org/wiki/Barberpole_illusion
Solving the ambiguity...


- How to get more equations for a pixel?
- **Spatial coherence constraint**
  - Assume the pixel’s neighbors have the same $(u,v)$
    - If we use a 5x5 window, that gives us 25 equations per pixel

\[
0 = I_t(p_i) + \nabla I(p_i) \cdot [u \ v]
\]

\[
\begin{bmatrix}
I_x(p_1) & I_y(p_1) \\
I_x(p_2) & I_y(p_2) \\
\vdots & \vdots \\
I_x(p_{25}) & I_y(p_{25})
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
= -
\begin{bmatrix}
I_t(p_1) \\
I_t(p_2) \\
\vdots \\
I_t(p_{25})
\end{bmatrix}
\]
Solving the ambiguity...

• Least squares problem:

\[
\begin{bmatrix}
I_x(p_1) & I_y(p_1) \\
I_x(p_2) & I_y(p_2) \\
\vdots & \vdots \\
I_x(p_{25}) & I_y(p_{25})
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
= -
\begin{bmatrix}
I_t(p_1) \\
I_t(p_2) \\
\vdots \\
I_t(p_{25})
\end{bmatrix}
\]

\[
A \ d = b
\]

\[
\begin{bmatrix}
\end{bmatrix}
\begin{bmatrix}
25\times2 & 2\times1 & 25\times1
\end{bmatrix}
\]
Matching patches across images

- Overconstrained linear system

\[
\begin{bmatrix}
I_x(p_1) & I_y(p_1) \\
I_x(p_2) & I_y(p_2) \\
\vdots & \vdots \\
I_x(p_{25}) & I_y(p_{25}) \\
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
\end{bmatrix}
= -
\begin{bmatrix}
I_t(p_1) \\
I_t(p_2) \\
\vdots \\
I_t(p_{25}) \\
\end{bmatrix}
\]

Least squares solution for \(d\) given by

\[
(A^T A) \ d = A^T b
\]

\[
\begin{bmatrix}
\sum I_x I_x & \sum I_x I_y \\
\sum I_x I_y & \sum I_y I_y \\
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
\end{bmatrix}
= -
\begin{bmatrix}
\sum I_x I_t \\
\sum I_y I_t \\
\end{bmatrix}
\]

\[
A^T A \\
A^T b
\]

The summations are over all pixels in the K x K window.
Conditions for solvability

Optimal \((u, v)\) satisfies Lucas-Kanade equation

\[
\begin{bmatrix}
\sum I_x I_x & \sum I_x I_y \\
\sum I_x I_y & \sum I_y I_y
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
= -
\begin{bmatrix}
\sum I_x I_t \\
\sum I_y I_t
\end{bmatrix}
\]

\(A^TA\)

\(A^Tb\)

When is this solvable? I.e., what are good points to track?

• \(A^TA\) should be invertible
• \(A^TA\) should not be too small due to noise
  – eigenvalues \(\lambda_1\) and \(\lambda_2\) of \(A^TA\) should not be too small
• \(A^TA\) should be well-conditioned
  – \(\lambda_1/\lambda_2\) should not be too large (\(\lambda_1\) = larger eigenvalue)

Does this remind you of anything?

Criteria for Harris corner detector
Low texture region

\[ \sum \nabla I (\nabla I)^T \]

- gradients have small magnitude
- small \( \lambda_1 \), small \( \lambda_2 \)
Edge

$$\sum \nabla I (\nabla I)^T$$

- large gradients, all the same
- large $\lambda_1$, small $\lambda_2$
High textured region

\[ \sum \nabla I (\nabla I)^T \]
- gradients are different, large magnitudes
- large \( \lambda_1 \), large \( \lambda_2 \)
The aperture problem resolved
The aperture problem resolved

Perceived motion
Errors in Lucas-Kanade

- A point does not move like its neighbors
  - Motion segmentation
- Brightness constancy does not hold
  - Do exhaustive neighborhood search with normalized correlation - tracking features – maybe SIFT – more later…

- The motion is large (larger than a pixel)
  1. Not-linear: Iterative refinement
  2. Local minima: coarse-to-fine estimation
Revisiting the small motion assumption

• Is this motion small enough?
  • Probably not—it’s much larger than one pixel
  • How might we solve this problem?
Coarse-to-fine optical flow estimation

Gaussian pyramid of image 1

Gaussian pyramid of image 2

u=1.25 pixels

u=2.5 pixels

u=5 pixels

u=10 pixels
Coarse-to-fine optical flow estimation

A. Bobick

image I

Gaussian pyramid of image 1

run iterative L-K

warp & upsample

run iterative L-K

image 2

Gaussian pyramid of image 2
Optical Flow Results

Lucas-Kanade without pyramids
fails in areas of large motion

* From Khurram Hassan-Shafique CAP5415 Computer Vision 2003
Optical Flow Results

Lucas-Kanade with Pyramids

* From Khurram Hassan-Shafique CAP5415 Computer Vision 2003