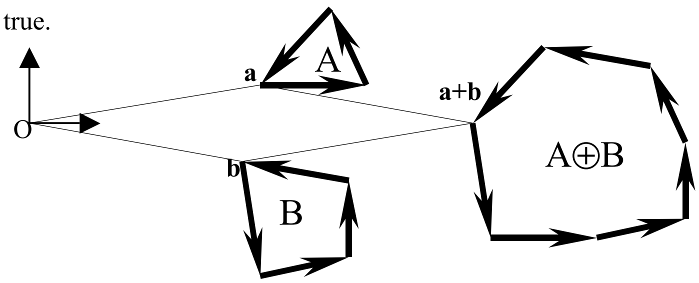
Distance measures

Herman **Minkowski** (1846-1909), Russian mathematician. Einstein's teacher. Developed geometric interpretation of relativity.
Felix **Hausdorff** (1868-1942), German mathematician. Founded a theory of abstract spaces on the notion of neighborhoods.
Maurice **Fréchet** (1878-1973), French mathematician. Originated the study of abstract spaces.

Notation

MIN{x: Y(x)} is the minimum of all x for which the predicate Y(x) is true. Similar definition for MAX.
□{X: Y} be the union of all sets X for which the predicate Y is true.
{**a**: P(**a**)} is the set of all points **a** for which the predicate P(**a**) is true.
A and B will be sets in 3D (unless specified otherwise).
a and **b** will always be two points such that **a**□A and **b**□B.
ab=**b**–**a** and **a+b**=**a**+(**b–o**).
o is the origin. Ø is the empty set. $\|\mathbf{v}\|^2=x_v^2+y_v^2+z_v^2$.

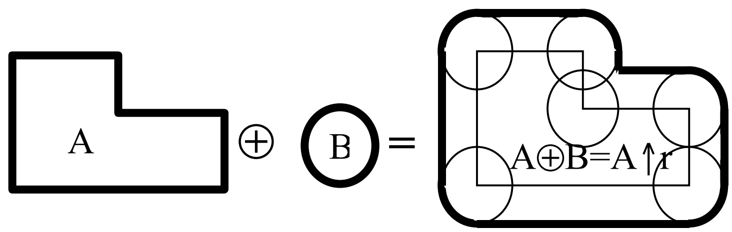


The Minkowski sumA⊕B is {a+b: **(**a,b**)□AxB}**.
The position of A⊕B depends on o, not the shape.

When A and B are convex polygons in two dimensions, then A⊕B is the convex polygon bounded by all their edges. It can be quickly constructed by orienting the edges of A and B counterclockwise, sorting them by angle, and using them in that order to build a closed chain.

Expansion

When B is a ball of radius r centered at the origin, A⊕B is the expansion A↑r of set A by a distance r.
Note that A↑r = □ {ball(**a**,r): **a**□A}, where ball(**a**,r) is a ball of radius r around a point **a**.



Minimum distance *(I love her and she loves me)*

The minimum distance M(A,B) is defined as MIN{||**ab**||: (**a**,**b**)□AxB}.
M(A,B)=M(B,A) and M(A,B) = MIN{r: (A↑r) □B≠ Ø}.
M(A,B) is the minimum distance by which you need to grow one of the sets so that it touches the other.
M(A,B) = 0 does not imply A=B, but A□B≠Ø □ M(A,B)=0
The minimum distance is useful to estimate safe travel between collisions, clearance between objects, swimming distance between islands, and minimum translation distance to establish contact between objects.
M(A,**b**) is the shortest distance from point **b** to set A.

Maximum deviation, also called the **Hausdorff distance** *(I love her and she hates me, or is it the reverse?)*

The Hausdroff distance: H(A,B) is defined as MIN{r: (B□(A↑r)) AND (A□(B↑r)) }.
Property or equivalent definition: H(A,B) = max(MAX{M(**a**,B): **a**□A} , MAX{M(**b**,A): **b**□B}).
H(A,B)=0 □ A=B, but A□B≠Ø does not imply H(A,B)=0.
The Hausdorff distance measures the discrepancy between two sets. It states how much you must travel to reach the furthest point of one set from the other. When H(A,B)<r, then we say that replacing A with B results in a Hausdorff error bounded by r.
Unfortunately, an arbitrarily small H(A,B) does not guarantee that A and B are similar.

Maximum displacement, also called the **Fréchet distance** *(For curves: dog on leash, no backtracking)*

The Fréchet distance F(A,B) is MIN{MAX{||**a**F(**a**)||: **a**□A}:F is an isomorphism (continuous bijection) mapping A to B}.
F(A,B)=F(B,A) and H(A,B) ≤ F(A,B)
The Fréchet distance is only valid when A and B have the same topology. Establishing the optimal mapping may be difficult.
It measures the minimim displacement distance between a point of A and its image in B.

Provide a high level description of the algorithms and of the geometric constructions for computing the following:
Problem 1: M(**a**,E), where E is the line segment from point **c** to **d.**
Problem 2: M(**a**,B), where **a** is a point and B is a solid bounded by a triangle mesh.
Problem 3: M(A,B), where A and B are two solids bounded by triangle meshes**.**
Problem 4: H(**a**,E), where E is the line segment from point **c** to **d.**
Problem 5: Explain how to compute H(A,B), where A and B are triangular faces in three space.
Problem 6: Let A and B be two Bezier curves. Let D(A,B) = MAX{||A(u)B(u)||: 0≤u≤1}. Compare D(A,B) to M(A,B), to H(A,B), and to F(A,B). Point out when one is never superior to the other. Provide simple examples when they are different.

CS6490 Homework 2 (Distance Measures) Individual work

SOLUTIONS

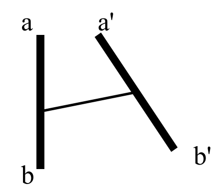
Let V_A, E_A, T_A respectively denote the collection of vertices, edges, and triangles of A. Same notation for B.

Problem 1 (5 points): M(**a**,E), where E is the line segment from point **c** to **d.**
*IF **ca**.**cd** and **da**.**cd** have opposite signs, THEN M(**a**,E)=||**ah**|| where **h**=**c**+(**ca**.**cd**)/||**cd**||, ELSE M(**a**,E)=MIN{||**ac**||, ||**ad**||}*

Problem 2 (10 points): M(**a**,B), where **a** is a point and B is a solid bounded by a triangle mesh.
<u>*Simple solution:*</u>
*IF **a**□B THEN M(**a**,B)=0 ELSE M(**a**,B)=MIN{M(**a**,T): T in T_B}.*
*IF the projection **h** of **a** on the plane of T falls on T, then M(**a**,T)=||**ah**||, ELSE M(**a**,T)=MIN{M(a,E): E in E_T, the edges of T}.*

<u>*A more efficient Solution:*</u>
*It **a**□B, return 0. Else compute d=MIN{||**av**||: v in V_B}. Then, for **c** and **d** consecutively bounding all the edges of B, test whether **ca**.**cd** and **da**.**cd** have opposite signs. If so, replace d with min(d,||**ah**||), where **h**=**c**+(**ca**.**cd**)/||**cd**||. Then, compute the projections of **a** onto the planes supporting triangles T of B. Each time a projection is closer to **a** than d, if it falls on T, update d.*

Problem 3 (15 points): M(A,B), where A and B are two solids bounded by triangle meshes**.**
If the first vertex of any connected component of A is inside B or vice versa, then M(A,B)=0.
*Otherwise, M(A,B)=min{MIN{M(**v**,B), **v** in V_A}, MIN{M(**v**,A), **v** in V_B}, MIN{M(e,e'), e in E_B, e' in E_A}}*



<u>*Solution for computing M(e,e'):*</u>
When the two edges are parallel, we reduce the problem to a 2D problem. Otherwise:
*Let N=(**ab**x**a'b'**)/||**ab**x**a'b'**||. It is the direction orthogonal to both lines, using the notation in the figure.*
IF (aa'.abxN and ab'.abxN have opposite signs AND a'a.Nxa'b' and a'b.Nxa'b' have opposite signs
*THEN M(e,e')=**aa'**.N) OTHERWISE M(e,e')=0.*

Problem 4 (5 points): H(**a**,E), where E is the line segment from point **c** to **d.**
*H(**a**,E)=max(||**ac**||,||**ad**||).*

Problem 5 (15 points): Explain how to compute H(A,B), where A and B are triangular faces in three space.
*H(A,B)=max(MAX{M(**a**,B): **a** in V_A}, MAX{M(**b**.A): **b** in V_B})*
*M(**a**,B) and M(**b**.A) are computed as in Problem 2.*

Problem 6 (10 points): Let A and B be two Bezier curves. Let D(A,B) = MAX{||A(u)B(u)||: 0≤u≤1}. Compare D(A,B) to M(A,B), to H(A,B), and to F(A,B). Point out when one is never superior to the other. Provide simple examples when they are different.

F(A,B) is never larger than D(A,B), because by definition, it is the minimum D(A,B) for all mappings.
F(A,B) is never smaller than H(A,B), because it is subject to an additional constraint.
Thus we have D(A,B) ≥ F(A,B) ≥ H(A,B) ≥ M(A,B)