Angular Momentum Computation for Multibody Systems

C. Karen Liu

School of Interactive Computing
Georgia Institute of Technology

1 Introduction

Computing angular momentum for a multibody dynamic system is very straightforward in maximal coordinates, where angular velocity of each rigid body is readily available. However, if you choose to use generalized coordinates and do not want to explicitly convert velocity to maximal coordinates at each time step, you can use the following alternative formulation to compute angular momentum.

2 Derivation

We begin with the computation of angular momentum for a single particle, $\mathbf{x}$. This is simply a cross product of its position vector from the pivot ($\mathbf{c}$) with its velocity ($\dot{\mathbf{x}}$), scaled by its mass μ

$$\mathbf{L} = \mu(\mathbf{x} - \mathbf{c}) \times \dot{\mathbf{x}} \tag{1}$$

Now let us consider the angular momentum of a multibody system with n rigid bodies connected in a hierarchical structure. We assume that the pivot is defined at the center of mass of the system $\mathbf{c}$, which could be varying over time ($\dot{\mathbf{c}} \neq \mathbf{0}$).

$$\mathbf{L} = \sum_{i=1}^{n} \iiint \mu_i(\mathbf{x}_i(x,y,z) - \mathbf{c}) \times (\dot{\mathbf{x}}_i(x,y,z) - \dot{\mathbf{c}}) dxdydz \tag{2}$$

where $\mathbf{x}_i(x, y, z)$ denotes a particle in i-th rigid body. μ_i is the infinitesimal mass of a particle in i-th rigid body, assuming each rigid body has uniform density.

This equation results in four terms:

$$\mathbf{L} = \sum_{i=1}^{n} \iiint \mu_i \mathbf{x}_i \times \dot{\mathbf{x}}_i - \sum_{i=1}^{n} \iiint \mu_i \mathbf{x}_i \times \dot{\mathbf{c}} - \mathbf{c} \times \sum_{i=1}^{n} \iiint \mu_i \dot{\mathbf{x}}_i + \sum_{i=1}^{n} \iiint \mu_i \mathbf{c} \times \dot{\mathbf{c}} \quad (3)$$

We omit the variables of integration (i.e. dx, dy, dz) for clarity. To compute Equation 3 efficiently, we need to rewrite it in a form without any integral. The first term is the most complicated one so we will deal with it later. The second, third and fourth terms can be simplified as follows:

Second Term: The integral of $\mu_i \mathbf{x}_i$ over the volume of i-th rigid body is equal to $m_i \mathbf{c}_i$, where m_i is the mass of i-th rigid body and $\mathbf{c}_i$ is its center of mass in the world coordinates. The second term can then be written as:

$$-\sum_{i=1}^{n} \iiint \mu_i \mathbf{x}_i \times \dot{\mathbf{c}} = -\sum_{i=1}^{n} m_i \mathbf{c}_i \times \dot{\mathbf{c}} = -m\mathbf{c} \times \dot{\mathbf{c}} \quad (4)$$

where m (without subscript i) is the total mass of the multibody system and $\mathbf{c}$ again is its center of mass in the world coordinates.

Third Term: Similarly, we can rewrite the integral part of the third term as $m_i \dot{\mathbf{c}}_i$ and arrive at a simpler third term:

$$-\mathbf{c} \times \sum_{i=1}^{n} \iiint \mu_i \dot{\mathbf{x}}_i = -\mathbf{c} \times \sum_{i=1}^{n} m_i \dot{\mathbf{c}}_i = -m\mathbf{c} \times \dot{\mathbf{c}} \quad (5)$$

Fourth Term: For this term, we integrate μ_i over each rigid body and sum up all the rigid bodies to obtain the following form:

$$\sum_{i=1}^{n} \iiint \mu_i \mathbf{c} \times \dot{\mathbf{c}} = m\mathbf{c} \times \dot{\mathbf{c}} \quad (6)$$

Combining these three terms, the angular momentum for the multibody system can be simplified to

$$\mathbf{L} = \sum_{i=1}^{n} \iiint \mu_i \mathbf{x}_i \times \dot{\mathbf{x}}_i - m\mathbf{c} \times \dot{\mathbf{c}} \quad (7)$$

The only remaining integral is in the first term. Before we work on the math, let us first introduce a new operator "cr()":

$$\mathrm{cr}(\mathbf{a}\mathbf{b}^T) = \mathbf{a} \times \mathbf{b} \quad (8)$$

where $\mathbf{a} \in R^{3\times 1}$ and $\mathbf{b} \in R^{3\times 1}$. Specifically, cr() takes a 3 by 3 matrix as input and outputs a 3 by 1 vector using the following rule:

$$\mathrm{cr}(A) = \begin{bmatrix} a_{23} - a_{32} \\ a_{31} - a_{13} \\ a_{12} - a_{21} \end{bmatrix}, \quad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{9}$$

With this new operator, we can rewrite the first term as follows:

$$\sum_{i=1}^{n} \iiint \mu_i \mathbf{x}_i \times \dot{\mathbf{x}}_i = \sum_{i=1}^{n} \iiint \mu_i \mathrm{cr}(\mathbf{x}_i \dot{\mathbf{x}}_i^T) \tag{10}$$

We can also express $\mathbf{x}_i$ in terms of its coordinates in the local frame of i-th rigid body: $\mathbf{x}_i = R_i \bar{\mathbf{x}}_i + \mathbf{r}_i$, where R_i and r_i are the rotation matrix and translation vector from the local frame of i-th rigid body to the world frame. $\bar{\mathbf{x}}_i$ denotes the local coordinates of the particle. Plugging this expression into Equation 10, we continue on our derivation of the first term:

$$\begin{aligned} \sum_{i=1}^{n} \iiint \mu_i \mathrm{cr}(\mathbf{x}_i \dot{\mathbf{x}}_i^T) &= \sum_{i=1}^{n} \iiint \mu_i \mathrm{cr}((R_i \bar{\mathbf{x}}_i + \mathbf{r}_i)(\bar{\mathbf{x}}_i^T \dot{R}_i^T + \dot{\mathbf{r}}_i^T)) \\ &= \sum_{i=1}^{n} \iiint \mu_i \mathrm{cr}(R_i \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i^T \dot{R}_i^T + R_i \bar{\mathbf{x}}_i \dot{\mathbf{r}}_i^T + \mathbf{r}_i \bar{\mathbf{x}}_i^T \dot{R}_i^T + \mathbf{r}_i \dot{\mathbf{r}}_i^T) \\ &= \sum_{i=1}^{n} \mathrm{cr}(R_i \iiint \mu_i \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i^T \dot{R}_i^T + m_i R_i \bar{\mathbf{c}}_i \dot{\mathbf{r}}_i^T + m_i \mathbf{r}_i \bar{\mathbf{c}}_i^T \dot{R}_i^T + m_i \mathbf{r}_i \dot{\mathbf{r}}_i^T) \end{aligned} \tag{11}$$

We denote $\bar{\mathbf{c}}_i$ as the center of mass of i-th rigid body in its own local frame. Note that the integral of the last three terms in Equation 11 are now expressed in terms of aggregated quantity $\bar{\mathbf{c}}_i$ and m_i. The only integral term remained is the first term. We then define M as the integral of outer product of $\bar{\mathbf{x}}_i$, which can be precomputed based on the shape of the rigid body.

$$M_i = \iiint \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i^T \tag{12}$$

Finally, we arrive at the simplified formula of angular momentum:

$$\boxed{\mathbf{L} = \textstyle\sum_{i=1}^{n} m_i \mathrm{cr}(R_i M_i \dot{R}_i^T + R_i \bar{\mathbf{c}}_i \dot{\mathbf{r}}_i^T + \mathbf{r}_i \bar{\mathbf{c}}_i^T \dot{R}_i^T + \mathbf{r}_i \dot{\mathbf{r}}_i^T) - m\mathbf{c} \times \dot{\mathbf{c}}} \tag{13}$$

3 Angular Momentum in Maximal Coordinates

For the sake of completeness, we also provide a formulation for computing angular momentum in maximal coordinates.

$$\mathbf{L} = \sum_{i=1}^{n} (m_i(\mathbf{c}_i - \mathbf{c}) \times (\dot{\mathbf{c}}_i - \dot{\mathbf{c}}) + R_i \bar{I}_i R_i^T \omega_i) \tag{14}$$

where $\bar{I}_i$ is the inertia matrix of i-th rigid body in the local frame, which can be precomputed based on the shape of the rigid body. ω_i is the angular momentum of i-th rigid body in the world frame. We can simplify the equation to make it look similar to Equation 13.

$$\boxed{\mathbf{L} = \textstyle\sum_{i=1}^{n} (m_i \mathbf{c}_i \times \dot{\mathbf{c}}_i + R_i \bar{I}_i R_i^T \omega_i) - m\mathbf{c} \times \dot{\mathbf{c}}} \tag{15}$$