Problem 1: Tail Inequalities, Chebyshev-Cantelli Bound

Let $X$ be a random variable with expectation $\mu_X$ and standard deviation $\sigma_X$.

(a) Show that for any positive real number $t$, \[ \Pr[|X - \mu_X| \geq t\sigma_X] \leq \frac{1}{1+t^2}. \] This version of the Chebushev inequality is sometimes referred to as the Chebyshev-Cantelli Bound.

(b) Prove that \[ \Pr[|X - \mu_X| \geq t\sigma_X] \leq \frac{2}{1+t^2}. \] Under what circumstances does this give a better bound than the Chebyshev inequality?

Problem 2: Tail Inequalities, Hoeffding’s generalized bound

(a) A function $f$ with domain and range the reals is said to be convex if for any $x_1$ and $x_2$ and $0 \leq \lambda \leq 1$, the following inequality is satisfied: \[ f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2). \] Show that the function $f(x) = e^{tx}$ is convex for any $t > 0$.

(b) Let $Z$ be a random variable that assumes values in the range $[0,1]$, and let $p = E[Z]$. Define the Bernoulli random variable $X$ such that $\Pr[X = 1] = p$ and $\Pr[X = 0] = 1 - p$. Show that for any convex function $f$, \[ E[f(X)] \leq E[f(Z)]. \]

(c) Let $Y_1, \ldots, Y_n$ be independent and identically distributed random variables over $[0,1]$, and define $Y = \sum_{i=1}^n Y_i$. Using parts (a) and (b) (or any other method you find helpful), show that, for some constant $c > 0$, \[ \Pr[|Y - E[Y]| > \Delta] \leq e^{-c\Delta^2/n}. \]

Problem 3: On Set Discrepancy

Let $\epsilon > 0$ be a constant. Show that there exists a constant $c = c(\epsilon)$ such that:

Where $(V,S)$ is a set system, where $|V| = n$ and $|S| = m$, as usual. Where $\chi$ be a random coloring of the elements of $V$, with each element mapped to +1 or -1, with probability 1/2 and independently from all other elements, as usual. Where we say that a set $S_i$ is bad if $|\chi(S_i)| > \sqrt{c|S_i|}$.

\[ \Pr[|\{i : S_i \text{ is bad}\}| > \epsilon m] < \frac{1}{2}. \]

Problem 4: MaxCut and Conditional Probabilities

(a) Let $G(V,E)$ be an undirected graph with $n$ vertices and $m$ edges. Partition $V$ to $A$ and $B$ (that is $A \cup B = V$ and $A \cap B = \emptyset$) and assign each vertex independently and equiprobably to either $A$ or $B$. Show that the expected number of edges that have one endpoint in $A$ and the other endpoint in $B$ is $m/2$.

(b) Use the method of conditional probabilities to obtain a deterministic polynomial time algorithm that finds a partition, or cut of $V$ to $A$ and $B$, such that the number of edges that have one endpoint in $A$ and the other endpoint in $B$ is at least $m/2$. 