- STRICT QUADRATIC PROGRAMS
  & their relaxations as VECTOR PROGRAMS
  (beyond LP-relaxations of IPs & towards SDP)

- A better than \( \frac{1}{2} \) approximation algorithm
  for MAX-CUT

- \( G(V, E) \) undirected graph
  edge weights \( w: E \rightarrow \mathbb{Q}^+ \)

Find partition \((S, \overline{S})\) of \( V \)
so as to maximize the total weight of edges in this cut
i.e., edges with one endpoint in \( S \) and the other in \( \overline{S} \).
\[(SQ)\]

\[
\max \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} \left(1 - y_i y_j \right)
\]

s.t. \(y_i^2 = 1\) \(\forall v_i \in V\)

\(y_i \in \mathbb{Z}\) \(\forall v_i \in V\)

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**Quadratic programs**: optimization & constraints are quadratic functions of integer valued variables

**Strict quadratic**: each monomial in the objective function and in the constraints is of degree 0 or 2

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For **MAX-CUT** in the way written above,

if \((S, \overline{S})\) is a partition

then \(v_i \in S \iff y_i = 1\) \(\forall v_i \in S\)

\(\iff y_i = -1\) \(\forall v_i \in \overline{S}\)
$$\begin{align*}
\text{(VP-relaxation)} \\
\max & \quad \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} \left(1 - \vec{v}_i \cdot \vec{v}_j \right) \\
\text{s.t. } & \quad \vec{v}_i \cdot \vec{v}_j = 1 \quad \forall \vec{v}_i \in V \\
& \quad \vec{v}_i \in \mathbb{R}^n \quad \forall \vec{v}_i \in V
\end{align*}$$

Vector Programs: optimize linear functions of inner products over vector variables subject to linear constraints of these inner products.

For MAX-CUT, the optimal solution (or a solution) to the above vector program assigns to each vertex a vector in $\mathbb{R}^n$. 
For MAXCUT each vertex is assigned to a vector in the unit sphere in $\mathbb{R}^n$.

The contribution of $\vec{v}_i$ and $\vec{v}_j$ in the optimization function is $\frac{w_{ij} (1 - \|\vec{v}_i\| \|\vec{v}_j\| \cos \theta_{ij})}{2} = \frac{w_{ij}}{2} (1 - \cos \theta_{ij})$.

The hope is that by spreading the vectors in $\mathbb{R}^n$ several / many ij will be reasonably far apart so that $\cos \theta_{ij} < 0$ or $\cos \theta_{ij} \leq 0$.

Clearly $\text{OPT(SQ)} \leq \text{OPT(VP)} = \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} (1 - \cos \theta_{ij})$.

Note Vector programs can be solved up to arbitrary "accuracy" $\varepsilon$ in time polynomial in $\varepsilon$.

For this lecture assume (VP) solved exactly in polynomial time.
**APPROX MAX-CUT**

1. Solve (VP) optimally and let $\vec{a}_1, ..., \vec{a}_n$ be an optimal solution.

2. Pick $\vec{r}$ uniformly distributed vector over unit sphere in $n$-dimensions.

3. $S = \{ v_i : \vec{a}_i \cdot \vec{r} \geq 0 \}$

4. $\overline{S} = \{ v_i : \vec{a}_i \cdot \vec{r} < 0 \}$

**THEOREM 1**

In expectation, the weight of edges crossing $(S, \overline{S})$ is at least 0.87856 $\text{OPT}(SQ)$.
Proof

\[ [W] = \sum_{1 \leq i < j \leq n} w_{ij} \Pr \left[ v_i \text{ and } v_j \text{ are separated} \right] \]

\[ \Pr \left[ v_i \text{ and } v_j \text{ are separated} \right] = \frac{\theta_{ij}}{\pi} \]

\[ \hat{a}_i \cdot \hat{r} \geq 0 \]

\[ \hat{a}_j \cdot \hat{r} \geq 0 \]

\[ \hat{a}_i \cdot \hat{r} < 0 \]

\[ \hat{a}_j \cdot \hat{r} < 0 \]

If the projection of \( \hat{r} \) on the \( \hat{a}_i, \hat{a}_j \) plane falls in these areas, then \( v_i \) and \( v_j \) are separated.
Thus \( E[w] = \sum_{1 \leq i < j \leq n} w_{ij} \frac{\Theta_{ij}}{\pi} \)

\[
= \sum_{1 \leq i < j \leq n} w_{ij} \frac{\Theta_{ij}}{\pi} \frac{1}{2} (1 - \cos \Theta_{ij}) \frac{1}{2} (1 - \cos \Theta_{ij})
\]

\[
= \sum_{1 \leq i < j \leq n} w_{ij} \frac{1}{2} (1 - \cos \Theta_{ij}) \left[ \frac{2}{\pi} \frac{\Theta_{ij}}{1 - \cos \Theta_{ij}} \right]
\]

Calculus: \( \frac{2}{\pi} \min_{0 \leq \Theta \leq \pi} \frac{\Theta}{1 - \cos \Theta} = \alpha > 0.87856 \)

\[
> \alpha \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij} (1 - \cos \Theta_{ij})
\]

\[
= \alpha \text{OPT (VP)}
\]

\[
> \alpha \text{OPT (SQ)}
\] end of proof
Approx Max-Cut

\[ e = \frac{0.8}{0.87856} \]

\[ c = \frac{e \alpha/2}{1 + e \alpha/2 - \alpha/2} \]

Run Algorithm on Page 5 \( \frac{1}{c} \) times
and output the heaviest cut found, say \( W' \)

Theorem 2 \[ \Pr \left[ W' \geq 0.8 \text{ OPT} \right] \geq 1 - \frac{1}{e} \]
Proof

\[ E[W] \leq \Pr \left[ W < (1-\varepsilon) E[W] \right] (1-\varepsilon) E[W] + \left(1 - \Pr \left[ W < (1-\varepsilon) E[W] \right] \right) \sum_{1 \leq i < j \leq n} w_{ij} \]

\[ \Rightarrow \]

\[ \Pr \left[ W < (1-\varepsilon) E[W] \right] \leq \frac{\sum_{1 \leq i < j \leq n} w_{ij} - E[W]}{\sum_{1 \leq i < j \leq n} w_{ij} - E[W] + \varepsilon E[W]} \]

\[ = 1 - \frac{\varepsilon E[W]}{\sum_{1 \leq i < j \leq n} w_{ij} + (\varepsilon - 1) E[W]} \]
\[ \sum_{1 \leq i < j \leq n} w_{ij} \geq E[W] \geq \alpha \text{OPT} \geq \frac{\alpha \sum_{1 \leq i < j \leq n} w_{ij}}{2} \tag{2} \]

Thus (1) \& (2) give

\[ \Pr \left[ W < (1-e)E[W] \right] \leq 1 - \frac{\epsilon \alpha}{2 \left( 1 + (e-1) \frac{\alpha}{2} \right)} \]

Thus

\[ \Pr \left[ W' > (1-e)E[W] \right] \leq (1 - c) \leq \frac{1}{e} \]

Thus

\[ \Pr \left[ W' \geq (1-e)E[W] \right] \geq 1 - \frac{1}{e} \]

and

\[ (1-e)E[W] \geq (1-e)e\text{OPT}, \] by choice of \( \epsilon \), so

\[ \Pr \left[ W' \geq 0.8\text{OPT} \right] \geq 1 - \frac{1}{e}. \]