1. Problem

(a) Undirected, weighted graph.
(b) Cut defined by set of vertices $S$, $w(\delta(S))$: weight of edges leaving $S$.
(c) Global mincut: find $S$ such that $S \neq \emptyset$, $S \neq V$ that minimize $\delta(S)$.
(d) Simple: pick one vertex $u$ in $S$, one vertex in $V \setminus S$, $v$. Compute minimum separating $u$ and $v$.
(e) $O(n^2)$ maxflow calls.

2. Structure of Cuts in Undirected Graphs (somewhat unrelated)

(a) Simple improvement: assume $u \in S$ by symmetry, only need to enumerate over $v$. $O(n)$ maxflow calls.
(b) Let $\delta(u, v)$ denote the min-cut separating $u$ and $v$.
(c) $\delta$ is an ultrametric, for any three vertices $u$, $v$, $w$, $\min\{\delta(u, v), \delta(v, w)\} \leq \delta(u, w)$.
(d) Proof: consider minimum cut separating $u$ and $w$. $v$ must be on the same side as either $u$ or $w$. So the same cut separate $v$ from either $u$ or $w$.
(e) Result: Gomory-Hu tree. Tree (usually not a subtree) where minimum cut between $u$ and $v$ are given the minimum weight edge on tree path between $u$ and $v$.

3. Simple, Randomized Algorithm for Minimum Cut

(a) Due to Karger, ’93.
(b) Randomly pick an edge with probability proportional to its weight. Contract it.
(c) Return cut given by final set of two vertices.
(d) Analysis: suppose $k$ vertices remain, minimum cut size $C$.
(e) Weight of edge incident to each vertex $\geq C$.
(f) Total weight of all edges $\geq kC/2$.
(g) Probability of an edge on minimum cut being chosen: $\leq k/2$.
(h) Success probability: $\frac{k-2}{k}$.
(i) Overall success probability: $\prod_i \frac{i-2}{i} = \frac{1}{(n-1)m}$.
(j) $\tilde{O}(n^2)$ iterations guarantees success w.h.p.
(k) All steps can be done in $\tilde{O}(m)$ time (uniformly sample all edges by weight, if end points are already contracted, remove edge from bag), $\tilde{O}(n^2m)$ algorithm.
(l) Further improvements to $\tilde{O}(n^2)$ by Karger, Stein ’96.

4. Other Useful Properties of Cuts

(a) Argument is made w.r.t. any $S$ defining a minimum cut, at most $O(n^2)$ such cuts.
(b) Similar: at most $O(n^{2\alpha})$ cuts of size $\alpha S$.
(c) Crucial result in construction of cut sparsifiers: graphs with $O(n \log n/\epsilon^{-2})$ edges that preserve weights of all cuts within multiplicative factors of $1 \pm \epsilon$. 