1. s, t Maximum Flow Problem
   (a) Directed graph, source s, sink t, capacities u, route maximum amount of flow.
   (b) Due to residual flows, more convenient to view as undirected graph where each edge has capacities in both directions. (edge = 2 arcs)
   (c) Goal: $\tilde{O}(m^{3/2}\log U)$ time, due to Goldberg-Rao ‘98.
   (d) Tools needed:
      i. Maxflow-mincut
      ii. Flow decompositions (implicit)
      iii. Residual graphs
      iv. Blocking Flows
   (e) Simplification (by binary search): final amount of flow routed is $F$.
   (f) If have flow $f$ that routes $F'$ units of flow, problem becomes equivalent to routing $F - F'$ units on graph with capacities $u - f$.
   (g) Equivalent goal: find flow $f$ with $F' \geq O(F/\sqrt{m})$ in $\tilde{O}(m)$ time.
   (h) Alternate view without knowing $F$:
      i. Maintain flow / cut, know value of flow is at most value of cut.
      ii. Decrease difference between value of cut and flow by factor of $1 - O(1/\sqrt{m})$.
      iii. LP view: decrease ‘gap’ between primal and dual solutions.

2. Algorithm
   (a) If arc has capacity more than $F/\sqrt{m}$, saturating it means we found a $f$ that we need, so can treat all such arcs as having infinite capacity.
   (b) Assign such arcs length 0, all other non-saturated arcs length 1, and saturated arcs length $\infty$. Let distance label be $l$.
   (c) Find distance (w.r.t $l$) of all vertices from $s$, $d_l$.
   (d) If $d_l(t) > \sqrt{m}$, then consider edges leaving all vertices with distance to $s$ at most $i$.
   (e) Each length 1 edge belongs to at most one such set.
   (f) Some set has $< \sqrt{m}$ edges, total capacities of edges leaving: $< F$, found smaller cut, contradiction to being able to route $F$ units of flow.
   (g) Otherwise, find blocking flow, either it saturates some length 0 edge that we contracted, or the entire blocking flow can be applied.
   (h) Later case increase distance of $t$ (w.r.t $l$) by 1. Can happen at most $O(\sqrt{m})$ times.

3. Issue 1: Blocking flow only works on DAGs.
   (a) Contract strongly connected components formed by length 0 edges
      i. Replace component with two spanning trees entering/leaving some vertex
ii. If total flow does not exceed $\frac{1}{4}(F/\sqrt{m})$, can route along trees.

(b) Sorting all vertices by distance and topological ordering in graph formed by length 0 arcs gives DAG to perform blocking flow calculation on.

4. Issue 2: going from $u$ to $u - f$ changes length function from $l$ to $l'$.

(a) Dual of shortest path w.r.t. $l$: maximize $d_l(t)$ over all valid distance labels.

(b) $d$ is a distance label for $l$ if $d_l(v) \leq d_l(u) + l(u \to v)$.

(c) Claim: $d_l$ is a valid distance label for $l'$. Proof: Arc $u \to v$ can only have length increase if flow routed from $v$ to $u$, but then $d_l(v) \leq d_l(u)$.

(d) Similar: show if blocking flow found, can increase distance labels on side unreachable from $s$ using DAG edges by 1 to get distance label for $l'$. 