1. Separators
   (a) Subset of vertices $X$ s.t. no connected components of $G \setminus X$ has more than $c|V|$ vertices.
   (b) Simplifying assumption: bounded degrees, vertex/edge separators are equivalent up to constants.
   (c) In general, vertex separator doesn’t imply edge separator. Example: star.
   (d) Set of vertices $S$, $\partial(S)$: edges leaving $S$, $N(S)$: vertices in $\bar{S}$ incident to $S$.
   (e) Bounded degree means $|\partial(S)|$ and $|N(S)|$ are about the same, work with $N(S)$.
   (f) $c$-balanced Separator: minimize $|N(S)|$ such that $c|V| \leq |S| \leq (1 - c)|V|$.
   (g) Key definition: conductance of cut defined by $S$:
   $$\Phi(S) = \frac{|N(S)|}{\min(|S|, |\bar{S}|)}$$
   (h) Goal today: give a brief overview of algorithms to find separators.

2. Sparsity Good
   (a) If have disjoint $S_1 \ldots S_k$ such that $\sum_i |S_i| \leq |V|/2$ and each $|S_i|$ has $|N(S_i)| \leq 1/l|S_i|$
   (b) $|N(\cup_i S_i)| \leq \sum_i |N(S_i)| \leq 1/l \sum_i |S_i| = |\cup_i S_i|$.
   (c) Algorithm for producing 1/4-balanced separator: find sparse $S_i$s until their total mass exceed 1/4$|V|$.
   (d) Can potentially go for $O(n)$ iterations.

3. Ball Growing
   (a) Conditional algorithm for finding a cut of small conductance.
   (b) Start with $S_0 = r$ for some vertex $r$.
   (c) If $|N(S_i)| < 1/l|S_i|$, stop with a small cut.
   (d) Else set $S_{i+1} \leftarrow S_i \cup N(S_i)$.
   (e) $|S_{i+1}| \geq (1 + 1/l)|S_i|$, can continue for at most $O(l \log n)$ steps.
   (f) If no cut found, get small diameter.
   (g) Problem: $|\bar{S}|$ is the one with smaller volume.
   (h) Fix: grow two layers at a time. Continue as long as the size of the ‘boundary’ is at least $1/l$ of either $|S|$ or $|\bar{S}|$.
   (i) Alternate view: continue as long as $\Phi(S_i) > 1/l$.
   (j) Either $|S|$ increase by $1 + 1/l$, or $|\bar{S}|$ decrease by $1 - O(1/l)$. Diameter certificate loose by factor of 2.

4. Vertex Separators on $K_h$ minor-free Graphs
(a) Special case: planar separators, $K_5$-free.
(b) Algorithm: $O(nm/l)$ time to find separators of size $O(n/l + 4lh^2 \log n)$, due to Plotkin-Rao-Smith ‘93.
(c) Maintain 3-way partition:
   ii. $V_{\text{done}}$: set of vertices on one side of the partition
   iii. $V_{\text{left}}: V \setminus M \setminus V_{\text{done}}$: what’s left to work on.
(d) Invariants:
   i. $|V_{\text{done}}| \leq 1/4|V|$.
   ii. $M$ can be partitioned into $A_1 \ldots A_k$ s.t. each piece has size $O(lh \log n)$, and any two pieces are adjacent. Equivalent to this partitioning giving a $K_k$ minor.
   iii. $|V_{\text{left}} \cap N(V_{\text{done}})| \leq 1/l|V_{\text{done}}|$.
(e) Final separator: $M \cup N(V_{\text{done}})$: $|M| + |V_{\text{done}}|/l \leq O(lh^2 \log n + n/l)$.
(f) Ball Growing on $V_{\text{left}}$:
   i. $A_1 \ldots A_k$: $k$ subsets of vertices.
   ii. Either has tree $T$ of depth $4l \log n$ s.t. $V(T) \cap A_i \neq 0$
   iii. Or $S \subseteq V_{\text{left}}$ with $\Phi(S) \leq 1/l$.
   iv. Note: exhaustion argument, no dependency on $k$.
(g) Algorithm: repeatedly do ball growing unless $k = h$.
   i. If some $A_i$ has no more neighbors in $V \setminus M \setminus V_{\text{done}}$, it can’t be part of a bigger minor, add it to $V_{\text{done}}$. (This is actually source of most of the work)
   ii. If tree produced, take one path to each $A_i$ with $i < k$ to create $A_k$.
   iii. Otherwise, has $S$ with small $\Phi(S)$. Add smaller half of this cut to $V_{\text{done}}$.
(h) Running time:
   i. A bit counterintuitive, bottleneck comes from creating the $A_k$.
   ii. Each set $S$ that we move to $V_{\text{done}}$ pays for itself.
   iii. Might spend $m$ time finding $A_k$, which is much smaller.
   iv. Invariant allows $A_k$ to have at least $l \log n$ vertices, pull extra ones from its connected component. If not enough, can directly move to $V_{\text{done}}$.
   v. Minimum progress made: $hl \log n$, can happen at most $n/(hl \log n)$ times.
   (i) Setting $l = \sqrt{n}/(h\sqrt{\log n})$ gives $O(h\sqrt{n} \log n)$ sized separators in $O(m^{3/2})$ time.
(j) Improvement: Wulff-Nilsen ‘11, $O(m^{5/4+\epsilon})$ time.

5. Approximately Optimal Sparsest Cut
   (a) Want to check: whether graph has cut of conductance at least $\alpha$.
   (b) $O(\log n)$ factor approximation: Leighton-Rao ‘89.
   (c) Main components: linear programming, multicommodity flow, and ball growing.
   (d) Arora-Rao-Vazirani ‘04: semi-definite programming, incorporates ideas from geometry.
   (e) Many improvements, state of the art: Sherman ‘09: $O(\sqrt{\log n})$ approximation using $O(n^\epsilon)$ single commodity flows.
   (f) Spielman-Teng ‘04: Local partitioning algorithms that produce cuts of ratio $\tilde{O}(\sqrt{\alpha})$. State of the art: OveisGharan-Trevisan ‘12.