18.434 Presentation Notes: Tutte matrices and maximum matchings

Let $G = (V, E)$ be an undirected graph, and let the vertex set be $V = \{1, 2, \ldots, n\}$.

Definition 1. *The* ***Tutte matrix*** T *of* G *is an* n *by* n *matrix with the* (i, j) *entry given by*

$$t_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \in E, i < j \\ -x_{j,i} & \text{if } (i,j) \in E, i > j \\ 0 & \text{otherwise} \end{cases}$$

where the $x_{i,j}$ *are independent variables.*

Remark 1. T *is skew-symmetric* ($T = -T^T$).

Let S_n be the symmetric group of order n, and for each $\sigma \in S_n$, let $\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}$, where $N(\sigma)$ is the number of inversions in σ, or the number of pairs (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$. Let $P_\sigma = \prod_{i=1}^n t_{i,\sigma(i)}$. We can write the determinant of the Tutte matrix as follows.

$$\det T = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) P_\sigma$$

Theorem 1. *G has a perfect matching if and only if* $\det T \neq 0$.

To prove the above theorem, we use a correspondence between permutations in S_n and directed cycles on V. In particular, each permutation $\sigma \in S_n$ such that $P_\sigma \neq 0$ corresponds to a disjoint union of directed cycles on V. For each $1 \leq i \leq n$, if $\sigma(i) = j$, we set a directed edge from i to j in the graph. Since $P_\sigma \neq 0$, each such edge (ignoring direction) is present in G. Moreover, each vertex has exactly one outgoing edge and one ingoing edge. Thus these directed edges form a disjoint union of directed cycles on the vertex set of G.

Lemma 1. *Let $O_n \subset S_n$ be the subset of permutations such that, in the corresponding set of directed cycles, at least one cycle is odd. Then* $\sum_{\sigma \in O_n} \operatorname{sgn}(\sigma) P_\sigma = 0$.

Proof. Let $\sigma \in O_n$. If the length of the longest odd cycle in σ has length 1, then $\sigma(i) = i$ for some i, which implies that $P_\sigma = 0$, thus this makes no contribution to the sum. If σ contains at least one odd cycle C of length $k \geq 3$. Let σ' be the same permutation as σ except with the direction of the edges in C reversed. Since σ, σ' have the same cycle structure, thus $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma')$. Since $P_\sigma = (-1)^k P_{\sigma'} = -P_{\sigma'}$, thus $\operatorname{sgn}(\sigma) P_\sigma + \operatorname{sgn}(\sigma') P_{\sigma'} = 0$, and their contributions to the sum cancel out. Since this applied for any permutation in O_n, thus the sum is zero as desired. □

Lemma 2. *G has a perfect matching if and only if there exists $\sigma \in S_n \backslash O_n$ such that* $P_\sigma \neq 0$.

Proof. We first assume that there exists $\sigma \in S_n \backslash O_n$ such that $P_\sigma \neq 0$. Note that σ corresponds to a disjoint union of directed cycles on V, and since $\sigma \notin O_n$, hence all the cycles are even. This gives a perfect matching in G by taking every other edge in these cycles.

Conversely, assume that G has a perfect matching given by the edges $(u_1, v_1), (u_2, v_2), \ldots (u_{n/2}, v_{n/2})$. n has to be even for G to be a perfect matching. This corresponds to a permutation $\sigma \in S_n$, $P_\sigma \neq 0$ such that $\sigma(u_i) = v_i$, $\sigma(v_i) = u_i$ for each i, and $\sigma \notin O_n$ as the corresponding directed cycles each have length 2. □

We now prove Theorem 1.

Proof. By Lemma 1, we have $\det T = \sum_{\sigma \in S_n \backslash O_n} \mathrm{sgn}(\sigma) P_\sigma$. If $\det T \neq 0$, then there exists some $\sigma \in S_n \backslash O_n$ such that $P_\sigma \neq 0$. By Lemma 2, G has a perfect matching.

Conversely, assume that G has a perfect matching. Then there exists some $\sigma \in S_n \backslash O_n$ such that $P_\sigma \neq 0$, as defined in the proof of Lemma 2. We can write $P_\sigma = (-1)^{n/2} \prod_{i \,:\, u_i < v_i} x^2_{u_i, v_i} \prod_{j \,:\, u_j > v_j} x^2_{v_j, u_j}$. No other term in the sum for $\det T$ contains the same set of variables, thus we have $\det T \neq 0$. □

For graphs that do not have perfect matchings, there is a more general result.

Theorem 2. *The rank of the Tutte matrix is equal to twice the size of the maximum matching.*

To prove this theorem, we use results from linear algebra.

Lemma 3. *A skew-symmetric matrix A of rank r has an r by r non-singular principal submatrix.*

Proof. We can assume that rows $1, 2, \ldots r$ of A are linearly independent. By skew-symmetry, columns $1, 2, \ldots r$ are also linearly independent. We claim that the r by r submatrix M formed by these rows and columns is nonsingular. This follows because if we assume to the contrary that M has rank less than r, we can do elementary row operations on the first r rows of A such that the first row of M is all zeroes. Since every column to the right of column r is a linear combination of columns 1 to r, it follows that the first row of A must be all zeroes, which contradictions the first r rows of A being linearly independent. The lemma follows from this claim. □

We now prove Theorem 2.

Proof. Let M be an r by r non-singular principal submatrix of T, and consider the subgraph H of G induced by the vertices corresponding to the rows or columns of M. Note that the rank r must be even, as a corollary to Lemma 3. Since $\det M \neq 0$, from Theorem 1, H has a perfect matching of size $r/2$. This is a matching in G.

Moreover, this matching is maximal in G. Suppose that there exists a matching of size greater than $r/2$, and consider the submatrix of T obtained by taking the rows or columns in T corresponding to vertices in this matching. This submatrix has size greater than r and nonzero determinant by Theorem 1. This implies that T has greater than r linearly independent rows, which is a contradiction. □