This test is online, posted via Canvas and the course homepage, and handed in via GradeScope. The GradeScope submission page will close at 1:30pm (Eastern Time), but we suggest that you start wrapping up by around 1:15pm. You have 135 minutes to earn up to $1 + 4 + 4 + 8 + 6 + 3 = 26$ points, the test is graded out of 25.

This booklet contains 5 questions on 7 pages, including this one.

Write your solutions in the space provided. If you run out of space, continue your answer on the back of the last page, and make a notation on the front of the sheet.

You may use any written or locally stored resources.

However, the only internet resources that you could use during the test are: specifically:

1. The BlueJeans Office Hours link at https://gatech.bluejeans.com/242024735, where clarifications will be posted in the chat.

2. Piazza Test 3 clarification page: https://piazza.com/class/k4xbfrttfnc687?cid=568

You may use any of the theorems/facts/lemmas from the lecture notes, homeworks, or textbook without re-proving them unless explicitly stated otherwise. However, we may check with you about the citation / reference afterwards, possibly via regrades.

If necessary make reasonable assumptions but please be sure to state them clearly

Do not spend too much time on any one problem. Generally, a problem’s point value is a good indication of how many minutes to spend on it.

Good luck!

0. ( /1 point) Submit your test through GradeScope before 1:30pm, under the right name/id, and with the pages corresponding to each problem clearly indicated.
1. (4 points) Give a pushdown automata for the following language:

$$L = \{ w \in \{0,1\}^* \mid \#0w \geq \#1w \},$$

that is, $$w$$ is the set of binary strings with more 0s than 1s.

SOLUTION:

![Diagram of pushdown automata]

GRADING:

- 1 marks for accepting number of 0's = number of 1's
- 2 marks for accepting number of 0's > number of 1's
- 1 point for marking start and accepting states
2. (4 points) Give a context free grammar for the language consisting of balanced parentheses in both ( ) and [ ]s. That is, the ( can be paired up with the )s, and the [s can be paired up with the ]s so that each substring between the corresponding pairs are also balanced parentheses. For example, [()()], [[[][]]) are balanced, while [()] is \textbf{NOT} balanced.

\textbf{SOLUTION:}

\[ s \rightarrow \epsilon \]
\[ s \rightarrow (s) \]
\[ s \rightarrow [s] \]
\[ s \rightarrow ss \]

\textbf{GRADING:}

- +1 for every correct rule (or equivalent)
3. ( 8 points) Prove that the following languages are not context free using Pumping Lemma for context free languages.

(a) ( 4 points)

\[ L = \{a^ib^jc^k \mid 0 \leq i < j < k \} . \]

**SOLUTION:** Take string \( a^pb^{p+1}c^{p+2} \) for pumping length \( p \). We have \( s \in L \) and \( |s| \geq p \). Now, by pumping lemma \( s \) must be partitioned into \( uvxyz \), with \( |vxy| \leq p \). We can divide it into two cases.

If \( vxy \) consist of only one type of characters, we can divide it into two subcases. First if it consists of \( a \) or \( b \), then after pumping up, the \( i < j < k \) condition won’t hold. Otherwise if it consists of \( c \), then after pumping down, the \( i < j < k \) condition won’t hold.

If \( vxy \) consist of two types of characters, we can divide it into two subcases. First if it consists of \( a,b \), then after pumping up, the \( i < j < k \) condition won’t hold. Otherwise if it consists of \( b,c \), then after pumping down, the \( i < j < k \) condition won’t hold. Because \( |vxy| \leq p \), it can’t consist of three types of characters at the same time.

**GRADING:**

- 1 point: Incorrect choice of string
- 1 point: Does not consider all cases of \( vxy \)
- 1 point: Choice of \( i \) is incorrect/not specific enough
- 1 point: Insufficient reasoning as to why \( uv^ixy^iz \notin L \)

(b) ( 4 points)

\[ L = \{0^n1^b2^a3^b \mid a, b \geq 0 \} . \]

**SOLUTION:** Take string \( s = 0^p1^p2^p3^p \) for pumping length \( p \). We have \( s \in L \) and \( |s| \geq p \). Now, by pumping lemma \( s \) must be partitioned into \( uvxyz \), with \( |vxy| \leq p \). So there are two cases for \( vxy \).

\( vxy \) consists of one type of digit: assume it contains all 1s, for other digits the proof is similar. In this case, since \( |vy| > 0 \), the string \( uv^ixy^iz \) is not in \( L \) for \( i \neq 1 \). Let \( n = |vy| \), then \( uv^ixy^iz = 0^n1^b+i2^a3^b \).

\( vxy \) spans at most two adjacent digit groups: assume the two groups are 0s and 1s, for other digits the proof is similar. Now at most 1 of \( v \) or \( y \) can contain both types of digits, assume it is \( v \). In this case we have \( v = 0^m1^n, y = 1^q \). Since \( |vy| > 0 \), we have \( m, n, q \geq 0, m+n+q > 0 \). In this case \( uv^ixy^iz \) is not in \( L \) for \( i \neq 1 \), as \( uv^ixy^iz = 0^{a+mi+nq}1^{b+qi}2^a3^b \).

**GRADING:**

- 1 point: Incorrect choice of string
- 1 point: Does not consider all cases of \( vxy \)
• -1 point: Choice of $i$ is incorrect/not specific enough
• -1 point: Insufficient reasoning as to why $uv^i xy^i z \notin L$
4. (6 points) Select and solve exactly two of the following four questions.

Unless you clearly indicate which two to mark, only the first two in lexicographical order with work on them will be marked.

For proving a language is context free, you may provide either a CFG or a PDA.

(a) Let
\[ L = \{ 0^i1^j \mid i \neq j, \; i, j \geq 0 \} . \]
Prove that this language is context free.

**SOLUTION:**

\[
S \rightarrow 0S1 \mid L \mid R
\]
\[ L \rightarrow 0L \mid 0 \]
\[ R \rightarrow R1 \mid 1 \]

(b) Let
\[ L = \{ a^i b^j c^k \mid i, j, k \geq 0, \; i + j = k \} . \]
Prove that this language is context free.

**SOLUTION:**

\[
S \rightarrow aSc \mid M
\]
\[ M \rightarrow bMc \mid \epsilon \]
GRADING:
- 3 marks for correct CFG
- 3 marks for correct PDA, -1 for not mentioning start and accepting states

(c) Let
\[ L = \{0^a1^b2^3^a \mid a, b \geq 0\} . \]
Prove that this language is context free.

SOLUTION:
\[
S \rightarrow 0S3 \mid T \\
T \rightarrow 1T2 \mid \epsilon
\]

GRADING:
- 3 marks for correct CFG
- 3 marks for correct PDA, -1 for not mentioning start and accepting states

(d) Consider the language over strings in a whose length is a factorial:
\[ L = \{a^n! \mid n > 0\} . \]
Here \( n! \) is the factorial of \( n \), defined as the product of integers from 1 to \( n \), \( n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 1 \). Prove that this language is not context free.

SOLUTION: Let \( p \) be the length provided by the CFG pumping lemma. Consider the string
\[ a^p! \]
and suppose it’s written as
\[ w = uv^ixy^iz \mid \text{for } i \geq 0 \]
with \( |vxy| \leq p \) and \( |vy| > 0 \).
Let \( |vy| = j \) where \( j \) is an integer between 1 and \( p \). When the string is pumped, it can be represented as \( a^{p+j} \). Then the string is not in the language because \((p + 1)! - p! = p(p!) > p \geq j \) so \( p! + j < (p + 1)! \). This suffices for when \( p > 1 \). Thus, the pumped string is not a factorial and not in the language \( L \).

GRADING:
- -1 bad choice of \( w \) (not in language or fixed length)
- -1 doesn’t consider all cases for \( vxy \) to arrive at \( a^{p+j} \)
- -2 insufficient reasoning for why \( uv^ixy^iz \notin L \) (NOTE: Saying “we will encounter a number of as between \( p! \) and \( (p+1)! \)” is NOT sufficient. You must prove this property and show that it is indeed never a factorial.)
5. (3 points) Provide a context free grammar for the following language

\[ L = \{0^i1^j0^k \mid j \geq k + i\} . \]

SOLUTION:

\[
\begin{align*}
S & \rightarrow LMR \\
L & \rightarrow 0L1 \mid \epsilon \\
R & \rightarrow 1R0 \mid \epsilon \\
M & \rightarrow M1 \mid \epsilon
\end{align*}
\]

GRADING:

- -2 The rules will generate string with \( j < k + i \) or some cases of \( j \geq k + i \) are not covered by the rules.
- -2 The format of the string is wrong, like generating ”10101”.
- -1 A simple modification to the rules can make the solution work. For example, some rules can’t generate ”1”.