Problem Set 11 (Version 0)

This problem set has a total of 4 problems on 2 pages. Written solutions should be submitted to
GradeScope before 9:00pm on Tuesday Dec 1.

Your solutions should be submitted according to the guidelines [https://www.cc.gatech.edu/~rpeng/CS4540_F20/ProblemSetGuidelines.pdf](https://www.cc.gatech.edu/~rpeng/CS4540_F20/ProblemSetGuidelines.pdf) In particular:

1. If you choose not to submit a typed write-up, please write neat and legibly.

2. No credit will be given to solutions obtained verbatim from the Internet or other sources, and
uploaded codes will be ran through similarity checking software.

==the first 2 problems can be solved using only the fact that the decision points of
the dynamic programs are monotonic

1. Given a sequence $n$ binary numbers, with $B$ digits each, find the fewest number of digits that
need to be changed to make them sorted, in $O(n^2B)$ time.

   Autograder: [https://dmoj.ca/problem/dmopc18c1p6](https://dmoj.ca/problem/dmopc18c1p6)

2. Given a sequence of $n$ non-negative integers $x[1...n]$, maximize the score over $k$ steps of the
following splitting process:

   (a) Take a sequence with length more than 1,
   (b) Split it somewhere in the middle to get 2 sequences,
   (c) Add score that’s the product of the sum of the two parts to the total.

   You should output the maximum score in $O(nk \log n)$ time or faster.

   Autograder: [https://dmoj.ca/problem/apio14p2](https://dmoj.ca/problem/apio14p2)

3. There are $n$ numbers $x[1...n]$ in a row, you start at some location $s$, and want to maximize the
a total number you collect over $k$ steps of:

   (a) move left, or right,
   (b) collect all numbers in the current location (that have not been collected before).
Note that although it’s possible to pass through the same location multiple times, the numbers there can only be collected once. Compute the maximum score in $O(n \log^2 n)$ time.

(Note: this needs both monotonicity and range queries)

Autograder: https://dmoj.ca/problem/ioi14p6

== the following two problems are about structure of points and lines

4. A duathlon is a race that involves running $r$ km and cycling $k$ km. $n$ contestants have entered the race; each contestant has different running and cycling speeds, $v_r[i]$ and $v_c[i]$ respectively. One of the contestants has bribed the organizers to set $r$ and $k$ in order to win by the maximum margin. Give a $O(n^5)$ time or faster algorithm for determine if this is possible, and if so the value of $r$ and $k$.

Autograder: https://dmoj.ca/problem/cc02p4

Note: we’re purposely being loose on the exact complexity needed, because there is actually an $O(n)$ time algorithm for this: https://sarielhp.org/teach/10/a_rand_alg/lec/29_lp_d.pdf.

5. Given $n$ points on the 2-D plane, and a sequence of $n$ non-vertical, infinite, lines $l[1]...l[n]$ (specified as $a[i]x + b[i]y + c = 0$ for some non-zero $y[i]$), for each point, find the earliest index $i$ where a line passes above or on that point, in $O(n^{1.5} \log n)$ time or faster. We are aware of an $O(n \log^2 n)$ time solution.

Autograder: https://dmoj.ca/problem/mmcc15p2

== the last three problems are about optimizing linear/convex functions

6. Given a list of $n$ numbers $c[1...n]$, along with a value $L$, partition the sequence into segments, minimize the total over all segments $l_k, r_k$ of

$$\left( r_k - l_k + \sum_{l_k \leq i \leq r_k} c[i] - L \right)^2$$

in $O(n \log n)$ time or faster.

Autograder: https://dmoj.ca/problem/ccoprep3p2

7. Given $n$ and $k$, find the maximum over a sequence of $k$ positive integers $x[1...k]$ that sum to $n$ of

$$\sum_{1 \leq i \leq k} \frac{x[i]}{\sum_{i \leq j \leq k} x[j]}$$

to additive accuracy $\epsilon$ in $O(n \log(n/\epsilon))$ time.

Autograder: https://dmoj.ca/problem/coci18c4p5