This problem set has a total of 10 problems on 2 pages. Written solutions should be submitted to Canvas by 2pm Wednesday Jan 20, 2020.

1. The angle between the hour and minute hands of a standard 12-hour clock is exactly 1 radian. Find all possible times in 12-hour format.

**SOLUTION:**

Based on Problem 1 from [https://www2.cms.math.ca/Competitions/CMO/archive/sol2003.pdf](https://www2.cms.math.ca/Competitions/CMO/archive/sol2003.pdf). Note however that version asked for number of solutions, instead of enumerating a list.
2. What’s the largest subset of divisors of 2021^{100} such that no element of this subset is a multiple of another element?

**SOLUTION:**
Consider the set $S = \{43^i 47^{100-i} : 0 \leq i \leq 100\}$. We claim that this is the good subset with maximal cardinality, $|S| = 101$.

We first show that no two elements, $s_a, s_b \in S$ are multiples are each other. Without loss of generality, let us first assume $s_a = 43^a 47^{100-a}$ divides $s_b = 43^b 47^{100-b}$.

This implies that $b \geq a$ and $100 - b \geq 100 - a$. Since we have $b \geq a$ and $a \geq b$, $s_a = s_b$ and by contradiction, no element in $S$ is a multiple of another.

Now we must show that $S$ is the largest possible subset. Consider a subset $S'$ that has more than 101 elements. Notice that any divisor of 2021^{100} with the form 43^a 47^b has only 101 powers
of 43 to choose from. By the pigeonhole principle, there must be two elements \( s_a, s_b \in S \) such that \( s_a = 43 \cdot 47^a \) and \( s_b = 43 \cdot 47^b \). Without loss of generality, we let \( b > a \) and, \( s_b \) is a multiple of \( s_a \). Thus, \( S \) is the largest possible subset.

3. How many ways can 8 rooks be placed on a \( 9 \times 9 \) chessboard so that none of them are attacking each other?

**SOLUTION:**

Note that there can only be 1 rook in each column and row of the chessboard. We will count the number of ways to place 9 rooks and then remove one.

Let us denote a valid placement of the 9 rooks with a permutation of “123456789”. The value of the \( i^{th} \) index of the string corresponds with the column of the rook in row \( i \). Thus, we have \( 9! \) ways of placing 9 rooks.

Notice that there are 9 ways of removing a rook from each placement of 9 rooks, resulting in \( 9! \times 9 = 3265920 \) ways to place 8 rooks on a \( 9 \) by \( 9 \) board.

4. Continue from problem (3), but the rooks have to all be on squares of the same color.

**SOLUTION:**

Without loss of generality let the corner tiles of the chessboard be black. We notice that if a rook is in a odd column, it cannot attack a rook on an even column if they are on the same color square.

First consider the case where the rooks are on black tiles. Notice that we can divide the problem into two chessboards, a \( 5 \) by \( 5 \) with just odd columns and a \( 4 \) by \( 4 \) with just even columns. Notice that the placement of rooks on each of these two boards can be done independently, since odd column rooks cannot attack even column rooks and vice versa. Placing 9 rooks and then removing one, we have \( 5!4! = 25920 \) placements.

Now consider the case where the rooks are on white tiles. This creates two chessboards, one of size \( 4 \) by \( 5 \) and the other of size \( 5 \) by \( 4 \). Notice that there must be 4 rooks on each board, since number of rooks \( \leq \) \( \min(\text{rows, columns}) \). Thus, we have \( 5! \times 5! = 14400 \) placements on white squares and \( 25920 + 14400 = 40320 \) total placements.

5. Consider a subset of \( \{1 \ldots 17\} \) with size 8. Prove that no matter what this subset is, there are three distinct pairs of elements from it whose differences are the same. For example, for the set \( \{1, 2, 3, 7, 9, 11, 15, 17\} \), we have \( 17 − 15 = 11 − 9 = 9 − 7 \). Note that pairs sharing one number are still different.

**SOLUTION:**


6. For a real number \( x \), let \( [x] \) denote the largest integer that’s at most \( x \). Find all solutions to

\[
x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{5} \right\rfloor
\]
SOLUTION:
The possible values of $x$ are:

\[ x = 0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, \]
\[ 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, 59 \]

7. Give a subset of 9 cells in a 6-by-6 square grid so that each selected cell is adjacent to a distinct number of unselected cells.

SOLUTION:
Select (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 5), (3, 3), (5, 4), (6, 5).

8. Suppose there are integers $x, y, z$ and $n$ such that

\[ x^2 + y^2 + z^2 - xy - yz - xz = n. \]

Show that we can also find integers $a$ and $b$ such that

\[ a^2 + b^2 - ab = n. \]

SOLUTION:
We want to find integers $a$ and $b$ in terms of $x, y, z$, so we have to manipulate the two expressions to match one another. The key to this problem is recognizing that the algebraic expression $(a - b)^2 = a^2 - 2ab + b^2$ is being hidden in the given and desired equations. In order to get the $2ab$ term to appear, we are motivated to multiply the desired equation by 2. This allows us to make the following simplifications to the desired equation of $a$ and $b$:

\[ 2a^2 + 2b^2 - 2ab = 2n \]
\[ a^2 + b^2 + (a^2 + b^2 - 2ab) = 2n \]
\[ a^2 + b^2 + (a - b)^2 = 2n \]

We do the same process of multiplying by 2 to the given equation of $x, y, z$:

\[ 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz = 2n \]
\[ (x^2 + y^2 - 2xy) + (x^2 + z^2 - 2xz) + (y^2 + z^2 - 2yz) = 2n \]
\[ (x - y)^2 + (x - z)^2 + (y - z)^2 = 2n \]

In this manner, we can see that setting $a = x - y$ and $b = x - z$ yields the desired expression. Note that $(a - b)^2 = (z - y)^2 = (y - z)^2$. 

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9. Show that every positive integer \( x \) that’s not a multiple of 20 has an integer multiple \( xy \) such that in the base 10 representation of \( xy \), every pair of adjacent digits have different parity.

**SOLUTION:**

10. For a parameter \( \lambda \) (that’s the focus of this problem), we start with \( n \) numbers, not all the same, and repeatedly perform the following moves an arbitrarily large number of times: pick two numbers \( a \) and \( b \) with \( a < b \), replace \( a \) with

\[
 b + \lambda (b - a) .
\]

Find all values of \( \lambda \) (as a function of \( n \)) such that for any initial set of numbers that are not all the same, there is a sequence of moves that leads to an arbitrarily large number.

In other words, for the \( \lambda \)s that work, you should give an algorithm that for any valid initial state and any threshold \( k \), produces a number that’s larger than \( k \).

**SOLUTION:**
Problem Set 1 (Version 3)

Updated: 10:40am, Sat Jan 23 2021  Due: 2:00pm, Wed Jan 27 2021

Change log:

• Problem 5 (version 3) clarified the order of evaluation of powers: the power inside the exponent is evaluated first.

• Problem 8 (version 2) sign flip on the 1.

This problem set has a total of 10 problems on 2 pages. Written solutions should be submitted to Canvas by 2pm Wednesday Jan 27, 2020.

• If you find the sources of these problems, a citation of the source is sufficient for full points.

• If you choose not to submit a typed write-up, please write neat and legibly.

1. Let \( d(n) \) be the number of divisors of \( n \). Show that the product of all divisors of \( n \) equals to \( n^{d(n)/2} \).

2. Prove for any integer \( n \), \( n^5 - n \) is divisible by 30.

3. Show that for any positive integer \( n \),

\[
\left\lfloor \frac{(n-1)!}{n^2 + n} \right\rfloor
\]

is always even.

**SOLUTION:**

4. Find the last three digits of \( 2021^{2020} \).

5. Find the last three digits of \( 2021^{2020^{2019}} \). Note that the power to 2019 is evaluated first: this is \( 2021 \wedge (2020 \wedge 2019) \), not \( (2021 \wedge 2020) \wedge 2019 \).

6. Show that for any odd prime \( p \),

\[
\sum_{i=1}^{p-1} i^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}.
\]
**Hint:** for $p = 11$, the values of $i^{2p-1} = i^{21}$ modulo $121 = 11^2$ are: 1, 101, 3, 37, 49, 61, 73, 107, 9, 109

**SOLUTION:**

7. Let $p \geq 5$ be a prime. Show that

$$\left(\frac{p^2}{p}\right) - p$$

is divisible by $p^5$.

**SOLUTION:**

8. Show that for any integer $n$, the number

$$10^{10^n} + 10^{10^n} + 10^n - 1$$

cannot be prime.

**SOLUTION:**

9. Find all integers $n > 1$ such that if $a$ and $b$ are relatively prime, then $a \equiv b \pmod{n}$ if and only if $ab \equiv 1 \pmod{n}$.

**SOLUTION:**
[https://prase.cz/kalva/short/soln/sh00n1.html](https://prase.cz/kalva/short/soln/sh00n1.html)

10. Find all natural numbers $n$ larger than 3 such that $2^{2000000}$ is divisible by

$$1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$$

**Note:** this can be solved with a lot of computers, but can also be done the Amish way...

**SOLUTION:**

It suffices to find $n$ such that

$$(n + 1) \cdot (n^2 - n + 6) = 3 \cdot 2^{k+1}$$

for some $k \geq 1$. Set

$$m \leftarrow n = 1$$

gives

$$m (m^2 - 3m + 8) = 3 \cdot 2^{k+1}.$$ 

Two cases:
(a) If \( m = 2^s \), then \( m^2 - 3m + 8 = 3 \cdot 2^t \). If \( s \geq 4 \), the LHS is 8 (mod 16), so \( t \leq 3 \). So either \( s \leq 3 \), or \( t \leq 3 \), checking by hand gives \( m = 8 \), or \( n = 7 \), is the only thing that works here.

(b) If \( m = 3 \cdot 2^u \), then \( m^2 - 3m + 8 = 2^v \). If \( v \geq 4 \), then \( 2^v \equiv 8 \) (mod 16), so we must have \( s \leq 3 \). Checking all possibilities there then gives \( s = 3 \) works, for \( n = 23 \).

So the answers are 7 and 23.