Change log:

- Problem 8, version 2: sign flip on the 1.

This problem set has a total of 10 problems on 2 pages. Written solutions should be submitted to Canvas by 2pm Wednesday Jan 27, 2020.

- If you find the sources of these problems, a citation of the source is sufficient for full points.

- If you choose not to submit a typed write-up, please write neat and legibly.

1. Let \( d(n) \) be the number of divisors of \( n \). Show that the product of all divisors of \( n \) equals to \( n^{d(n)/2} \).

**SOLUTION:**

Let \( d_1, d_2, \ldots, d_{d(n)} \) be the divisors of \( n \). We can exploit the symmetry of divisors to simplify our product. Note that if \( d_i \) is a divisor, then \( n/d_i \) is also a divisor. There is a one-to-one correspondence between these two, hence:

\[
\left( \prod_{i=1}^{d(n)} d_i \right)^2 = \prod_{i=1}^{d(n)} d_i \prod_{i=1}^{d(n)} \frac{n}{d_i}
\]

\[
= \prod_{i=1}^{d(n)} d_i \prod_{i=1}^{d(n)} \frac{n}{d_i}
\]

\[
= \prod_{i=1}^{d(n)} n
\]

\[
= n^{d(n)}
\]

So to get the desired product, we take the square root to get that

\[
\prod_{i=1}^{d(n)} d_i = n^{d(n)/2}.
\]
2. Prove for any integer $n$, $n^5 - n$ is divisible by 30.

**SOLUTION:**
We can think of this problem in terms of the Chinese Remainder Theorem (CRT), if $n^5 - n$ is divisible by 30 then it is also divisible by the primes 2, 3, and 5 simultaneously.

**METHOD 1:** By (an alternative formulation of) Fermat’s Little Theorem on primes 2, 3, 5, we have that for all integers $n$:

$$
\begin{align*}
  n^2 - n &\equiv 0 \pmod{2} \\
  n^3 - n &\equiv 0 \pmod{3} \\
  n^5 - n &\equiv 0 \pmod{5}
\end{align*}
$$

And it follows that $n^2 - n = n(n-1)$ and $n^3 - n = n(n-1)(n+1)$ both divide $n^5 - n = n(n-1)(n+1)(n^2 + 1)$. Hence, $n^5 - n$ is divisible by 2, 3, and 5.

**METHOD 2:** We can factor the term as follows:

$$
n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = n(n-1)(n+1)(n^2 + 1)
$$

We know that 2 divides one of $(n-1), n,$ or $(n+1)$ since any 2 consecutive numbers will be 0, 1 (mod 2) in some order.

Similarly, we know that 3 divides one of $(n-1), n,$ or $(n+1)$ since any 3 consecutive numbers will be 0, 1, 2 (mod 3) in some order.

Finally, we can do casework on $n$ (mod 5). If $n \equiv 1, 0, 4 \pmod{5}$, then $(n-1), n,$ and $(n+1)$ will be 0 (mod 5) respectively. Else, $n \equiv 2$ or 3 (mod 5), which implies that $n^2 + 1 \equiv 0 \pmod{5}$. Hence, 5 always divides $n^5 - n$.

Since 2,3, and 5 divide $n^5 - n$, it is divisible by $2 \cdot 3 \cdot 5 = 30$ as well.

3. Show that for any positive integer $n$,

$$
\left\lfloor \frac{(n-1)!}{n^2 + n} \right\rfloor
$$

is always even.

**SOLUTION:**


4. Find the last three digits of $2021^{2020}$.

**SOLUTION:**

This is equivalent to finding $2021^{2020} \pmod{1000}$. We have that $(2021, 1000) = 1$ and $\phi(1000) = 400$, so by Euler’s Theorem:

$$
2021^{400} \equiv 1 \pmod{1000}
$$
Hence,
\[ 2021^{2020} \equiv 2021^{5 \cdot 400 + 20} \pmod{1000} \]
\[ \equiv (2021^{400})^5 \cdot 2021^{20} \pmod{1000} \]
\[ \equiv 21^{20} \pmod{1000} \]

From here we can just do exponential via repeated squaring, or alternatively, use the binomial theorem:
\[ 21^{20} \equiv (20 + 1)^{20} \equiv \binom{20}{2} 20^2 + \binom{20}{1} 20^1 + \binom{20}{0} 20^0 \equiv \boxed{401} \pmod{1000} \]

5. Find the last three digits of $2021^{2020^{2019}}$.

**SOLUTION:***

From Euler’s Theorem, we know that $2021^{400} \equiv 1 \pmod{1000}$. So we want to find what the exponent $2020^{2019}$ is mod 400. We have that:
\[ 2020^{2019} \equiv 20^{2019} \equiv 400 \cdot 20^{2017} \equiv 0 \pmod{400} \]

Since the exponent is 0 (mod 400), we have that $2021^{2020^{2019}} \equiv 2021^{400k} \equiv 1 \pmod{1000}$.

6. Show that for any odd prime $p$,
\[ \sum_{i=1}^{p-1} i^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}. \]

**Hint:** for $p = 11$, the values of $i^{2p-1} = i^{21}$ modulo 121 = $11^2$ are: 1, 101, 3, 37, 49, 61, 73, 107, 9, 109

**SOLUTION:**

7. Let $p \geq 5$ be a prime. Show that
\[ \binom{p^2}{p} - p \]
is divisible by $p^5$.

**SOLUTION:**
8. Show that for any integer \( n \), the number

\[
10^{10^n} + 10^{10^n} + 10^n - 1
\]
cannot be prime.

**SOLUTION:**

9. Find all integers \( n > 1 \) such that if \( a \) and \( b \) are relatively prime, then \( a \equiv b \pmod{n} \) if and only if \( ab \equiv 1 \pmod{n} \).

**SOLUTION:**
[https://prase.cz/kalva/short/soln/sh00n1.html](https://prase.cz/kalva/short/soln/sh00n1.html)

10. Find all natural numbers \( n \) larger than 3 such that \( 2^{2000000} \) is divisible by

\[
1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.
\]

**Note:** this can be solved with a lot of computers, but can also be done the Amish way...

**SOLUTION:**

It suffices to find \( n \) such that

\[
(n + 1) \cdot (n^2 - n + 6) = 3 \cdot 2^{k+1}
\]

for some \( k \geq 1 \). Set

\[
m \leftarrow n + 1
\]
gives

\[
m \left( m^2 - 3m + 8 \right) = 3 \cdot 2^{k+1}.
\]

Two cases:

(a) If \( m = 2^s \), then \( m^2 - 3m + 8 = 3 \cdot 2^t \). If \( s \geq 4 \), the LHS is 8 (mod 16), so \( t \leq 3 \). So either \( s \leq 3 \), or \( t \leq 3 \), checking by hand gives \( m = 8 \), or \( n = 7 \), is the only thing that works here.

(b) If \( m = 3 \cdot 2^u \), then \( m^2 - 3m + 8 = 2^v \). If \( v \geq 4 \), then \( 2^v \equiv 8 \pmod{16} \), so we must have \( s \leq 3 \). Checking all possibilities there then gives \( s = 3 \) works, for \( n = 23 \).

So the answers are 7 and 23.