DISCLAIMER: These notes are not necessarily an accurate representation of what I said during the class. They are mostly what I intend to say, and have not been carefully edited.

The plan for today is to introduce singular value decompositions and utilize them to give a fast 2-approximation to the $k$-means problem. The next sequence of lectures will deal with matrices, for which there is a good set of notes from previous offerings of this courses at http://www.cc.gatech.edu/~vempala/spectralbook.pdf. As a result, these notes will be mostly short summaries.

1. Singular values:

(a) When $A$ is not symmetric, cannot hope for $Av = \lambda v$ due to dimension mismatch.

(b) Instead use $A^T$ to bring the vector back to the same dimension:

\begin{align}
Av &= \sigma u \\
A^T u &= \sigma v.
\end{align}

(c) Combined: $A^T A v = \sigma^2 v$.

(d) Note: if $A$ is symmetric, eigenvectors of $A^2$ same as eigenvectors of $A$, $\sigma = \lambda$.

(e) Property: if we restrict $v$ to unit vectors, $\|v\| = 1$, then

\[ \|Av\|_2^2 = \sum_{1 \leq i \leq m} \|A_{(i)}\|_2^2 - d(A_{(i)}, v)^2, \]

where $A_{(i)}$ is the ith row of $A$, and $d(A_{(i)}, v)$ is its distance to the space spanned by the vector $v$.

2. Singular value decomposition:

(a) Define iteratively as $\max_{\|v_1\|=1} \|Av_1\|_2^2$, $\ldots$, $\max_{\|v_i\|=1, v_i \perp v_j \forall j < i} \|Av_i\|_2^2$.

(b) $V_k$: space spanned by $v_1 \ldots v_k$.

(c) Can show: $V_k$ minimizes

\[ \sum_{1 \leq i \leq m} dist(A_{(i)}, V). \]

over subspaces $V$ of dimension $k$. 
(d) Furthermore, the projected point is given by:

\[ \sum_{i=1}^{k} \sigma_i u_i v_i^T. \]

3. \( k \)-Variance Problem:

(a) Find a set of \( k \) points \( B^{(1)} \ldots B^{(k)} \in \mathbb{R}^n \) to minimize

\[ \sum_{i=1}^{m} \min_{1 \leq j \leq k} \| A^{(i)} - B^{(j)} \|_2^2 \]

(b) Exact algorithm: \( O(m^{k^2d/2}) \) by enumerating hyperplanes: each separating plane defined by \( d \) points.

(c) Faster algorithm: take SVD, then solve with \( A_k \).

(d) Can show via sequence of triangle inequalities that this gives a 2-approximation.