DISCLAIMER: These notes are not necessarily an accurate representation of what I said during the class. They are mostly what I intend to say, and have not been carefully edited.

Last time we introduced the singular value decomposition, and used it to give faster algorithms for k-means clustering. The plan today is to use it to give good approximations to MAX-2-SAT, or more generally MAX-2-CSP. The generalization to MAX-r-CSP also leads to the notion of low rank tensor approximations

• 2-SAT:
  – Constraints of the form $x_i \lor x_j$, or $\neg x_i \lor x_j$.
  – Find assignment to $x_i$ maximizing the number of satisfied constraints.
  – Algebraic formulation: let vector

    \[ y = [x_1; x_2; \ldots x_n; 1 - x_1; 1 - x_2; \ldots; 1 - x_n], \]

    and $A$ be matrix that has 1 in each corresponding term. Then the objective to maximize is

    \[ y^T A y \]

• Density condition:
  – Node weight: $d_i$: number of constraints that the ith literal appears in.
  – average degree:

    \[ \bar{d} = \frac{1}{2n} \sum_i d_i. \]

  – Core strength:

    \[ \sum_{ij} \frac{A_{ij}^2}{(d_i + \bar{d})(d_j + \bar{d})} \]

    – High core strength case: nearly uniform, e.g. almost every constraint is present.
    – Goal: $O(\epsilon n \bar{d})$ approximation for when core-strength is $O(1)$.

• Algorithm:
  – define the rescaling matrix $D_{ii} \overset{\text{def}}{=} \sqrt{d_i + \bar{d}}$, compute

    \[ B \leftarrow D^{-1} A D^{-1}, \]
– Set \( k \leftarrow O(1/\epsilon^{-2}) \).
– Build \( B_k \) using SVD.
– Solve on \( \tilde{A} \overset{\text{def}}{=} DB_kD \) approximately.

\[ \cdot \text{Error from going to the SVD:} \]

\[ y^T(\tilde{A} - A)y = y^TDB_kDy. \]
– Matrix inequality: \( x^T M x \leq \|x\|_2 \|Mx\|_2 \leq \|x\|^2_2 \|M\|_2 \).
– 2-norm of matrix: \( \|M\|_2 = \max_x \|Mx\|_2 / \|x\|_2 \), same as maximum singular value.
– Lemma: \( \|B - B_k\| \leq \frac{1}{\sqrt{k}} \|B\|_F \), proof: \( \|B\|_F = \|\sigma\| \), where \( \sigma \) is the vector of all singular values.
– Since all entries of \( y \) are \( \pm 1 \), \( \|Dy\| \leq \sum_i d_i + \bar{d} = 4n\bar{d} \).
– Overall error: \( 4n\epsilon \bar{d} \cdot (\text{core strength})^{1/2} \).

\[ \cdot \text{Solving low rank case} \]

– Write \( B_k \) as \( U \Sigma V \).
– \( yDB_kDy \) becomes \( u^T \Sigma v \) where \( u = UDy \), and \( v = VDy \).
– Key observation: \( u \in \mathbb{R}^k! \)
  * All entries of \( \Sigma \) are \( O(1) \) by assumption of core-strength.
  * Can ‘bucket’ entries of \( u \) and \( v \) into \( \epsilon \) sized buckets with overhead of \( \epsilon^{-O(k)} \).
– Check if each bucket of \( u \) is feasible.
– Integer program:
  1. \( 0 \leq y_i \leq 1 \).
  2. \( y_i + y_{i+n} = 1 \).
  3. \( u_i \leq (UDy)_i \leq u_i + \epsilon \), where \( u_i \) is approximately the value for the current ‘bucket’.
– Relax to fractional solution: \( k \) dimensions means if solution \( y \) exists, there is one with \( O(k) \) non-integral points.
– Perturbation caused by changing \( k \) points of \( y \) by at most 1

\[ y^T DADy \leq \|A\|_2 \|Dy\|_2^2, \]

if this set is \( S \), then this is bounded by \( \sum_{i \in S} (\bar{d} + d_i) \).
– Fix: give ‘special treatment’ to indices \( i \) with \( d_i > \Omega(\epsilon^2 n\epsilon \bar{d} k^{-2}) \).
– Only \( O(k^2 \epsilon^{-2}) = O(\epsilon^{-6}) \) such points, can brute force the values of the \( y_i \) entries with an extra overhead of \( \exp(\epsilon^{-6}) \).