DISCLAIMER: These notes are not necessarily an accurate representation of what I said during the class. They are mostly what I intend to say, and have not been carefully edited.

Last Thursday we discussed random walks on graphs. Today we start to relate them back to linear algebraic objects via graph Laplacians, and discuss their relation with effective resistances, which are Rayleigh quotients against the graph Laplacian matrix.

1 Escape Probabilities

The discussion of escape probabilities last time can be abstractly viewed as the following problem: for a random walk starting at \( u \), find its probability of reaching \( t \) before it reaches \( s \).

For this, we can define a probability \( p_u \) that denotes these values for each vertex \( u \). The boundary condition gives:

\[
\begin{align*}
p_s &= 0 \\
p_t &= 1
\end{align*}
\]  

while at intermediate vertices, we have that \( p_u \) is the weighted average of its neighbors, aka. it is a harmonic function:

\[
p_u = \sum_{v \sim u} \frac{w_{uv}}{d_u} p_v,
\]

or if we move the \( d_u \) term to the left, we get:

\[
d_u p_u = \sum_{v \sim u} w_{uv} p_v.
\]

To quickly check this, consider the path from 0 to \( n \), with \( s = 0 \) and \( t = n \). Then it can be checked that

\[
p_i = \frac{i}{n}
\]

is a solution that meets these conditions. Therefore as \( n \to \infty \), we’re highly likely to come back to the origin in 1D.
2 Solving for Escape Probabilities

Recall the graph Laplacian matrix of an undirected, weighted graph $G$, $L(G)$:

- Diagonals: weighted degree, $L_{uu}^{(G)} = \sum_{vv} w_{uv}^{(G)}$.
- Off-diagonals: negations of edge weights, $L_{uv}^{(G)} = -w_{uv}^{(G)}$.

Note that the condition of

$$d_u p_u - \sum_{v \sim u} w_{uv} p_v = 0$$

is exactly encoded as $L_u p = 0$ for each $u$ that’s not $s$ or $t$.

Furthermore, given such a vector, there are exactly two more degrees of freedoms: adding the all 1s vector to it so that $p_s = 0$, and rescaling it so that the difference between $s$ and $t$ is exactly 1.

So one way to find such a vector is to solve the system

$$L x = \chi_{s,t},$$

where $\chi_{s,t}$ is ‘two point’ indicator vector:

$$\chi_{s,t}(u) \overset{\text{def}}{=} \begin{cases} -1 & \text{if } u = s \\ 1 & \text{if } u = t \\ 0 & \text{otherwise} \end{cases}.$$

Then we normalize this vector so that the difference between $t$ and $s$ is 1,

$$\frac{L^T \chi_{s,t}}{\chi_{s,t}^T L^T \chi_{s,t}}.$$

Then since $p_s = 0$, the value of $p_u$ is the difference between the $u$th and $s$th entry in this vector, aka.

$$\frac{\chi_{s,u} L^T \chi_{s,t}}{\chi_{s,t}^T L^T \chi_{s,t}}.$$

Here we use the pseudoinverse because the null space of $L_G$ of a connected graph is the all 1s vector. One way to see this is via the Rayleigh quotient,

$$x^T L x = \sum_{u \sim v} w_{uv} (x_u - x_v)^2.$$

Which means if we have $L x = 0$, we must have $x_u = x_v$ for all $u \sim v$. More generally, the rank of $L_G$ is the number of connected components of it.
3 Bounding Escape Probabilities via Effective Resistances

Recall that the escape probability from $u$ to $t$, while avoiding $s$, is

$$\frac{\chi_{s,u} L^\dagger \chi_{s,t}}{\chi_{s,t} L^\dagger \chi_{s,t}}.$$

Matrix Cauchy Schwarz inequality gives:

$$\chi_{s,u} L^\dagger \chi_{s,t} \leq \sqrt{\chi_{s,u} L^\dagger \chi_{s,u} \chi_{s,t} L^\dagger \chi_{s,t}}$$

The value $\chi_{s,t} L^\dagger \chi_{s,t}$ is useful because it is numerator of a Rayleigh quotient. This particular form actually has a natural interpretation: if we let each edge $e$ be a resistor with conductance $w_e$, aka. resistance $1/w_e$, then this value is the ‘effective resistance’ between $u$ and $v$. This value is quite important, and sometimes I’ll denote it by the short hand $r_{uv}$. It obeys all the natural properties of resistors in series and in parallel:

1. Two resistors in series, $r_1$ and $r_2$ has effective resistance $r_1 + r_2$.
2. Two resistors in parallel has conductance $w_1 + w_2$, or resistance

$$\frac{1}{\frac{1}{r_1} + \frac{1}{r_2}}.$$

With this definition in mind, the above inequality is:

$$\chi_{s,u} L^\dagger \chi_{s,t} \leq \sqrt{r_{s,u} r_{s,t}},$$

so to show that this value tends to 0, it suffices to show that

$$\frac{r_{s,u}}{r_{s,t}} \to 0.$$

On the 1D path with $u = 1$, we have $r_{0,1} = 1$, and $r_{0,n} = n$ by resistance in series. So this gives yet another proof that we’re unlikely to escape from 0 on the 1D path.

Next time we will formalize this connection to resistances, and do similar arguments for the 2D and 3D cases.