DISCLAIMER: These notes are not necessarily an accurate representation of what I said during the class. They are mostly what I intend to say, and have not been carefully edited.

Logistical items:
1. Problem Set 3 is out.
2. Problem Set 2 is due at end of next week.
3. On Thursday Peng (the TA) will teach about sparsification based linear system solvers.

The main goal today is to discuss iterative methods, which are ways of solving systems of linear equations through a sequence of matrix-vector multiplications.

1 The General Idea: Good Polynomials

Solving the system of linear equations

\[ Ax = b \]

can be viewed as a ‘division’ by \( A \): \( x \leftarrow A^{-1}b \).

However, division is usually much more expensive than multiplications, especially involving matrices. So it’s useful to think about what happens with scalars, say we want to approximate \( a^{-1} \) without divisions, then one possibility is to use the identity

\[ \frac{1}{1-t} = 1 + t + t^2 + t^3 \ldots, \]

which in this case becomes

\[ \frac{1}{a} = 1 + (1-a) + (1-a)^2 + \ldots. \]

Note that converges when \(|1-a| < 1\). Furthermore, in this case, the error after evaluating the first \( i \) terms is

\[ \frac{(1-a)^i}{a}, \]

which means that if we’re guaranteed that \( a \) is bounded away from 0 (and 2), e.g.

\[ 0.1 \leq a \leq 1.9, \]
then this relative error of this method is bounded by $0.9^i$. This in turn implies that to get a relative error of $\epsilon$, we need to just evaluate $O(\log(1/\epsilon))$ terms of such a sequence.

The spectral theorem means that this holds for matrices as well. If we have

$$0.1I \preceq A \preceq I,$$

then the above argument can be applied to each of the eigenvalues separately to show that

$$A^{-1} \approx I + (I - A) + (I - A)^2 + \ldots + (I - A)^i.$$ 

Furthermore, in general, if we know that the minimum and maximum eigenvalues of $A$ are bounded by $\lambda_{\min}$ and $\lambda_{\max}$ respectively, then we can divide $A$ by the scalar $\lambda_{\max}$ to obtain $A'$ such that:

$$\frac{\lambda_{\min}}{\lambda_{\max}} I \preceq A' \preceq I,$$

which gives an iteration count bound of $O(\kappa \log(1/\epsilon))$.

## 2 Polynomial Evaluation, and Matrix Based Proofs of Convergence

Evaluating $A^i x$ takes $i$ matrix-vector multiplications involving $A$, so the above algorithm appears to take time $O(m \log^2(1/\epsilon))$. However, note that the power series $1 + t + t^2 + \ldots$ factors into $1 + t(1 + t(1 + t))$, which means that it can be instead viewed as the recurrence

$$x(i) \leftarrow b + (I - A) x(i-1) = x(i-1) + (b - Ax(i-1)).$$

This recurrence gives another interpretation of iterative methods: we compute the ‘error’ of the solution $x(i-1)$ at each step, which if formally termed the residue, and add it to the current iterand. Formally we define the residue as:

$$r(i) \overset{\text{def}}{=} b - Ax(i).$$

The convergence of these iterations can in turn be measured in terms of the residues. Substituting the recurrence:

$$x(i) = x(i-1) + (b - Ax(i-1))$$

into the recurrence gives:

$$r(i) = b - Ax(i)$$

$$= b - Ax(i-1) - Ar(i-1)$$

$$= (I - A) r(i-1),$$

(1) (2) (3)
upon which induction (or iteratively applying this) and substituting in the initial point of $x^{(0)} = 0$ gives

$$ r^{(i)} = (I - A)^i b. $$

So once again, we get that after $O(\kappa \log(1/\epsilon))$ iterations, the error is bounded by

$$ \| r^{(i)} \|^2_2 = \left\| (I - A)^i b \right\|^2_2 = b^T (I - A)^{2i} b \leq \left( 1 - \frac{1}{\kappa} \right)^{2i} \| b \|^2_2. $$

It’s also worth mentioning that the error is often specified in the $A$-norm, for the exact solution $x = A^{-1} b$, the error of a current solution $x$ is given in

$$ \| x - \bar{x} \|_A = \| r \|_{A^{-1}}, $$

and the error is usually measured relative to $\| b \|_{A^{-1}} = \| \bar{x} \|_A$.

### 3 Preconditioned Iterative Methods

Often instead of having $A$ that’s close to $I$, we instead have access to a matrix $B$ such that

$$ A \preceq B \preceq \kappa A. $$

In this case, we can modify the iterative method to solve the system

$$ B^{-1} Ax = B^{-1} b $$

instead, at which point the iterative method will give the recurrence:

$$ x^{(i)} \leftarrow x^{(i-1)} + (B^{-1} b - B^{-1} A x^{(i-1)}) = x^{(i-1)} + B^{-1} (b - A x^{(i-1)}), $$

aka we multiply the residue by $B^{-1}$ when adding a step.

This has some connections with mirror descent. However, because we’re working with matrices and systems of linear equations, we can directly work on the operators. Tracking the residue gives:

$$ r^{(i)} = b - A x^{(i)} $$

$$ = b - A x^{(i-1)} - B^{-1} A r^{(i-1)} $$

$$ = (I - B^{-1} A) r^{(i-1)}. $$

So in general if we have $B$ such that

$$ \| I - B^{-1} A \| \leq 1 - \frac{1}{\kappa} $$

then $O(\kappa \log(1/\epsilon))$ iterations suffices for getting an error of $\epsilon$ in this norm.
Note that the norm is intentionally left ambiguous: any $\ell_2$-like norm extended to operators work. For an operator $Z$, its norm is given by
\[
\|Z\| \overset{\text{def}}{=} \max_x \frac{\|Zx\|}{\|x\|},
\]
and we’re completely free in choosing the norm of the vector, as long as $\|I - B^{-1}A\|$ is small. If we restrict to matrix norms, we also have that if $P \approx \kappa Q$, then
\[
\|A\|_P \approx \kappa \|A\|_Q.
\]

4 Existence of a Good Matrix Norm

We now show that there is a sane norm. Specifically the condition of
\[
A \preceq B \preceq \kappa A,
\]
implies
\[
\|I - B^{-1}A\| \leq 1 - \frac{1}{\kappa}.
\]

The second condition is equivalent to:
\[
(I - B^{-1}A)^T A (I - B^{-1}A) \preceq \left(1 - \frac{1}{\kappa}\right)^2 A,
\]
or if we symmetrically compose both sides by $A^{-1/2}$:
\[
A^{-1/2} (I - B^{-1}A)^T A (I - B^{-1}A) A^{-1/2} \preceq \left(1 - \frac{1}{\kappa}\right)^2 I.
\]

Note that composing by $A^{1/2}$ on the left and $A^{-1/2}$ on the right is a similarity transform, which means that
\[
A (I - B^{-1}A) A^{-1/2} = I - A^{1/2} B^{-1} A^{1/2},
\]
while the condition of
\[
A \preceq B \preceq \kappa A,
\]
becomes
\[
\frac{1}{\kappa} A^{-1} \preceq B^{-1} \preceq A^{-1}, \quad \tag{7}
\]
\[
\frac{1}{\kappa} I \preceq A^{1/2} B^{-1} A^{1/2} \preceq I, \quad \tag{8}
\]
\[
0 \preceq I - A^{1/2} B^{-1} A^{1/2} \preceq \left(1 - \frac{1}{\kappa}\right) I. \quad \tag{9}
\]

Squaring the last one then gives the result.
5 Iterative Refinement

Note that iterative refinement only requires access to $B^{-1}$, instead of the matrix $B$. As a result, a useful notion that comes out of iterative methods is the notion of a ‘linear algorithm’. Specifically, as long as an algorithm does not do branches based on the values of the matrices, it can be written as a linear operator $Z$. In the case of a solver, it can be viewed as a black box as doing

$$x \leftarrow Zb,$$

where $b$, the RHS, is the ‘input’.

Then the notion of an approximate solver is one that produces

$$\|Zb - b\|_{A^\dagger} \leq \epsilon \|b\|_{A^\dagger}$$

for any vector $b$.

Expanding this out shows that this is actually equivalent to

$$Z^\dagger \approx A,$$

so we can invoke iterative methods with $B = Z^\dagger$ to give black-boxed methods for improving the accuracies of solvers. Specifically if we have an 0.1-approximate solver, invoking it $O(\log(1/\epsilon))$ times leads to a solver with $\epsilon$-error.

Iterative refinement more or less says that it suffices to solve linear systems to constant factor accuracy. It can also be used to turn a good solver in $P$ norm into a good solver in $Q$ norm.