• DISCLAIMER: These notes are not necessarily an accurate representation of what I said during the class. They are mostly what I intend to say, and have not been carefully edited.

• Graph Decompositions
  – Motivation: divide-and-conquer on graphs.
  – For simplicity: unweighted, undirected graphs.
  – Goal: cut $\beta m$ edges to form ‘nicer’ pieces, examples:
    * Line: cut every $\beta^{-1}$ edge.
    * Complete graph: can’t cut.
  – ‘Nice’ pieces: most widely used is well-connected pieces.
    * $E(S, V \setminus S)$: set of edges leaving $S$, $vol(S)$: total degree in $S$.
    * If $E(S, V \setminus S) \leq \beta vol(S)$, put $S$ into its own cluster.
    * Repeat until termination.
    * Can ensure: size of $S$ halves at each step.
    * Charge each cut edge to the volume of piece: $\beta \log nm$ edges cut.
  – Problem: finding min conductance cut is NP-complete. Alternate, simpler to compute decompositions.

• Low diameter decompositions: diameter of pieces bounded by $O(\beta^{-1} \log n)$.
  – Distributed computing: diameter directly related to computation steps.
  – Spanners: $O(n)$ edges so that all original edges have endpoints within $O(\log n)$.
    * From decomposition: join each vertex to all neighboring components.
    * Analysis similar to that of cutting probability: vertex adjacent to $O(1)$ clusters in expectation when $\beta = 1$.

• Existence of low diameter decompositions
  – Start with $S_0 = \{u\}$ for some arbitrary $u$.
  – If $E(S_i, V \setminus S_i) \leq \beta vol(S_i)$, stop.
  – Else $S_{i+1} \leftarrow S_i \cup N(S_i)$ ($N(S_i)$: neighbors of $S_i$).
- Bound on number of steps:
  * \( \text{vol}(S_{i+1}) \geq \text{vol}(S_i) + E(S_i, V \setminus S_i) \geq (1 + \beta)\text{vol}(S_i) \).
  * \((1 + \frac{1}{k})^k \approx e.\)
  * Get a number bigger than \(m\) in \(O(\beta^{-1} \log n)\) steps.
- Used in many algorithms (next class: turn graphs into trees).
- Issue: highly sequential process.

- More symmetric / parallel algorithm:
  - Need: shortest path tree contained in each piece (strong diameter).
  - Tie break based on distance.
  - Direct tie breaking rule: assign \(u\) to \(\arg \min_v \delta_v + d(u, v)\).
  - If \(u\) assigned to \(v\), entire shortest path from \(u\) to \(v\) assigned to \(v\).
  - Implementation: BFS to distance about \(\max(\delta_u)\), parallel if all \(\delta_u\) are small.

- Picking \(\delta_v\):
  - Intuition: set graph on fire:
    * Compete graph: 1 vertex burn everything
    * Line: about \(\beta n\) fires.
    * Interpolate: gradually set the graph on fire.
  - Choice: \(\delta_u \leftarrow -\text{Exp}(x, \beta)\):
    * Density at \(x\): \(\beta \exp(-\beta x)\).
    * Cumulative at \(x\): \(1 - \exp(-\beta x)\).
  - One vertex goes on fire at \(-\beta \log n\), two more at \(-\beta (\log n - 1)\) etc...

- Implementing this Decomposition (covered during Lecture 4 on Aug 27):
  - Breath first search (BFS) / Dijkstra’s algorithm:
    * Finds shortest paths from a single vertex to all vertices.
    * Visit vertices in increasing order of distance: can because edge weights are non-negative
    * Each edge visited once, \(O(m \log n)\) time with priority queue (can optimize, see PS1 problem 7).
    * Parallelization: process one layer (nodes with same distance) at a time.
  - Create super source, \(s^*\).
  - Distance from \(s^*\) to \(u\): \(\delta_u\).
  - Negative lengths! Fix by setting length from \(s\) to \(u\) to \(\delta_u - \min_v \delta_v\).
Fractional distances: all other edges are integral, fractional part only serves as tie-breaking.

- **Analysis**
  
  - Maximum diameter of a piece:
    * $u$ is a candidate for $u$, dist $-\delta_u$.
    * W.h.p. $\delta_u \leq O(\log n\beta^{-1})$ for all $u$ by union bound.
  
  - Probability of each edge being cut:
    * $e$ cut if two fires reach its midpoint within 1 time unit of each other.
    * View from midpoint ($m$): exponential decays starting at times $-\text{dist}(m, u)$.
    * Memoryless property of exponential distributions:
      \[
      \Pr_X [X \geq n + m | x \geq m] = \Pr_X [X \geq n].
      \]
    * Difference between largest and second largest still follows $\text{Exp}(x, -\beta)$.
    * Probability of fire in 1 unit of time: $1 - \exp(-\beta) \approx \beta$. 