• **DISCLAIMER:** These notes are not necessarily an accurate representation of what I said during the class. They are mostly what I intend to say, and have not been carefully edited.

• What’s a good tree? Criteria that you probably know:
  - Minimum / maximum spanning tree
  - Shortest path tree

• Representative graphs to consider: complete graph, cycle, square mesh. Issues:
  - Some edge must do badly.
  - Large portions of the graph is allowed to do badly
  - Fix: do well 'on average.

• Stretch of an edge $uv$ of $G$ w.r.t. a graph $H$:
  - Unweighted: $\text{dist}_H(u, v)$.
  - Weighted: $\frac{\text{dist}_H(u, v)}{\text{dist}_G(u, v)}$.
  - Spanner: minimize maximum stretch

• Low stretch trees: $T$ s.t.
  - spanning tree / embeddable into $G$ / doesn’t take shortcuts.
  - $\sum_e \text{stretch}(e) \leq m \log^c n$.

• Low stretch spanning tree on $\sqrt{n} \times \sqrt{n}$ square grid
  - Diameter: $\sqrt{n}$, some edge must have stretch $\sqrt{n}$.
  - Recursive $C$ construction.
  - $i$ edges with stretch $n/i$ for all $i = 2^j$, total: $O(n \log n)$.

• Bartal’s Algorithm
  - For simplicity: unweighted
  - Idea: emulate the tradeoff above.
* Decompose into pieces of diameter \( n, n/2, n/4, \ldots, 2^{-i}n, \ldots, 1 \).
* When diameter is \( d \), cut edges lead to stretch \( O(d) \).
  * Need: cut about \( m/d \) edges when decomposing into pieces of diameter \( d \).

  - Recall low diameter decompositions: diameter \( d \), probability of edge being cut \( \leq O(\log n/d) \).
- \textbf{BartalDecompose}(\( G, d \)):
  * Low diameter decompose into pieces of diameter \( d/2 \).
  * Recurse on all pieces.
  * Connect centers together via a shortest path tree.

- Key lemma: stretch of edge cut by piece with diameter \( d \): \( O(d) \).
  * Centers = Steiner vertices
  * Walk up the centers
  * Total distance: \( 2(d + d/2 + d/4 + \ldots) = O(d) \).

- Expected stretch of an edge

\[
\sum_{d=2^i} O(d) \frac{\log n}{d} = O(\log^2 n).
\]

  - Embeddable: lose another factor of \( O(\log n) \). Stronger: spanning trees, [AN'12]: \( O(m \log n \log \log n) \). Weaker: trees don’t shorten distances, [FRT '03]: \( O(\log n) \) expected per edge.

- Implementing Low Diameter Decompositions (moved to Notes for Lecture # 3, Aug 25)