• DISCLAIMER: These notes are not necessarily an accurate representation of what I said during the class. They are mostly what I intend to say, and have not been carefully edited.

• Summary
  – General relaxation via Log barriers.
  – Central path and optimality condition

• Problem: general linear programming:

\[
\begin{aligned}
\min_{x} & \quad c^T x, \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0.
\end{aligned}
\]

• Central path:
  – Relax with log-barriers:

\[
\min_{Ax=b} c^T x - 1^T \log x.
\]
  – Central path: a collection of these,

\[
\Phi(t)(x) = t \cdot c^T x - 1^T \log x.
\]

• Path following interior point method:
  – Start with \( t_0 \) tiny, and \( x \) that’s very close to optimum for \( \Phi(t_0) \).
  – Each step, perform few ball-constrained minimization steps to reduce \( \Phi(t_i)(x) \).
  – Then increase \( t \): \( t' \leftarrow \left( 1 + \frac{1}{O(\sqrt{m})} \right) t \).
  – \( t \) doubles every \( O(\sqrt{m}) \) steps, get convergence in \( O(m^{1/2} \log(U)) \) steps, where \( U \) is maximum magnitude of an entry (universe size).
  – Key invariant: at any point, \( x \) is nearly optimal for the current problem.

\[
\phi(t)(x) \leq \min \phi(t) + O(1).
\]
Note: this plus fast linear system solvers give $\tilde{O}(m^{3/2}\log(U))$ time algorithms for almost every single-commodity flow problems.

- Newton steps / ball constrained minimization / second order step:
  - Gradient: $\nabla(t) = t \cdot c - x^{-1}$.
  - Still need to move within null space of $A$, step:
    $$\max_{\Delta} (\nabla(t))^T \Delta$$
    $$A \Delta = 0$$
    $$\|X^{-1} \Delta\|_2 \leq 1$$
  - Dual:
    $$\min_y \|X (A^T y - \nabla(t))\|_2 = \min_y \|X (Ay - t \cdot c) + 1\|_2$$

- Progress:
  - From last time: if progress is $\alpha$, can obtain $f(t)(x') \leq f(t)(x) - 0.5\alpha$.
  - Main claim this time: if we have $y$ s.t. $\|X (Ay - t \cdot c) + 1\|_2 \leq 0.1$, then the current $x$ is within 0.2 of the optimum of $\Phi(t+1)$.
  - Consequence (modulo starting condition): $O(1)$ Newton steps after each adjustment of $t$ (by $1 + 1/\sqrt{m}$).

- Dual: $\max_y b^T y$ subject to $A^T y \leq c$.
  - Complementary slackness: $\langle x^*, c - A^T y^* \rangle = 0$.
  - Log barrier optimum:
    $$X^* (c - A^T y^*) = \frac{1}{t} 1.$$  
  - Algorithm is essentially controlled approach toward complementary slackness.

- Technical proof: fancier version of weak-duality:
  - Treating ‘slack’ vector $c - A^T y = s$ as a separate entity:
    $$\phi(t')(x) = t' c^T x - \log(x)$$
    $$= t' (A^T y + s) x - \log(x)$$
    $$= t' (A^T y)^T x + t' s^T x - \log(x).$$
  - Each $t's_i x_i - \log(x_i)$ minimized at $1/t's_i$.
  - Lower bound for $\phi(t')$: $t' (A^T y)^T x + m - 1^T \log(t's)$.
Last step: bound gap between $\phi^{(t')} (x)$ and this lower bound:

$$
t' c^T x - 1^T \log x - t' (A^T y)^T x - m - 1^T \log(t's)
= t' (c - A^T y)^T x - 1^T \log (t' X s)
= t' x^T s - 1^T \log (t' X s) - m.
$$

- Let $z_i = x_i s_i$.
- We have $\|tz - 1\|_2 \leq 0.2$.
- Triangle inequality: $\|t'z - 1\|_2 \leq t'/0.2 + (1 - t'/t) \|1\|_2 = 0.3$.
- This also means $0.7 \leq t' z_i \leq 1.3$.
- First order behavior of $\log(\alpha)$ when $0.7 \leq \alpha \leq 1.3$:
  $$\log(1 - \alpha) \geq -\alpha - 10 \cdot \alpha^2$$
- Substituting in $\alpha = 1 - t' z_i$ gives:
  $$\log(t' z_i) \geq t' z_i - 1 - 10(1 - t' z_i)^2.$$ 
- Summing allows us to bound the duality gap by:
  $$10 \sum (1 - t' z_i)^2 = 10 \|t' z - 1\|_2^2 \leq O(1).$$