Laplacians are Complete for Linear System over $\mathbb{Z}_p$

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Abstract

We show that for any prime number $p$, solving general Laplacian linear systems over the finite field $\mathbb{Z}_p$ is as hard as solving general linear systems over the finite field $\mathbb{Z}_p$, and for small prime number $p$, solving unit weight Laplacian linear systems over the finite field $\mathbb{Z}_p$ is as hard as solving general Laplacian linear systems over the finite field $\mathbb{Z}_p$. These results show that, over the finite field $\mathbb{Z}_p$, it’s unlikely to design faster solver specific to graph-structured linear systems that run faster than solvers for general systems.

1 Introduction

Solving a linear system $Ax = b$ efficiently is a fundamental problem in algorithm design. Efficient algorithms for solving linear systems have found broad applications in scientific computing, engineering and physics. There have been extensive research in how to solve general linear systems efficiently [ST11, Woo14, DMM08, LG14]. Besides solving general linear systems, recently much progress on faster solvers on graph-structured solvers has been made [Gre96, KMP12, ST14, JS20]. Over the reals, the current state-of-art solver for graph-structured solver achieves nearly linear time [JS20], which is faster than the fastest solver for general linear systems which requires $O(n^w)$ with $w < 2.372864$ [LG14].

Although there has been a large amount of effort in designing efficient algorithms for solving general or graph-structured linear systems over the reals, there is relatively less research in solving general or graph-structured linear systems over the finite field $\mathbb{Z}_p$. The finite field case is important for several reasons: in the general case, they are useful for producing exact fractional solutions [Dix82], while in the graph case they arise from network encoding [CLL11], which can in turn be used to compute edge/vertex connectivities. Therefore, it’s natural to ask, over the finite field $\mathbb{Z}_p$, is solving graph-structured linear systems still faster than solving general linear systems? Specifically, do Laplacian linear systems or unit weight Laplacian linear systems for undirected graphs have a faster solver than general linear systems?

In this paper, we present two simple reductions to show the hardness of solving graph-structured linear systems over $\mathbb{Z}_p$. The first one is from solving general linear systems to
solving Laplacian linear systems over the finite field $\mathbb{Z}_p$, which shows solving Laplacian linear system is as hard as solving general linear systems over $\mathbb{Z}_p$. The formal result is the following:

**Theorem 1.1.** For any prime $p$, $A \in \mathbb{Z}_p^{n \times n}$ and $b \in \mathbb{Z}_p^n$, solving a linear system $Ax \equiv b \mod p$ can be reduced to solving a Laplacian system $Ly \equiv c \mod p$ with $\text{nnz}(L) = O(\text{nnz}(A))$.

The second reduction is from solving general Laplacian linear systems to solving unit weight Laplacian linear systems over the finite field $\mathbb{Z}_p$, which shows for small prime number $p$, solving unit weight Laplacian linear systems is as hard as solving general Laplacian linear systems. We proved the following theorem:

**Theorem 1.2.** For any prime $p$, Laplacian matrix $L \in \mathbb{Z}_p^{n \times n}$ and $c \in \mathbb{Z}_p^n$, solving a Laplacian linear system $Lx \equiv c \mod p$ can be reduced to solving a unit weight Laplacian system $\hat{L}\hat{y} \equiv \hat{c} \mod p$ with $\text{nnz}(\hat{L}) = O(p \cdot \text{nnz}(L))$.

The above two theorems imply:

**Corollary 1.3.** For any prime $p$, $A \in \mathbb{Z}_p^{n \times n}$ and $b \in \mathbb{Z}_p^n$, solving a general linear system $Ax \equiv b \mod p$ can be reduced to solving a unit weight Laplacian system $\hat{L}\hat{y} \equiv \hat{c} \mod p$ with $\text{nnz}(\hat{L}) = O(p \cdot \text{nnz}(A))$.

These results imply unlike over the reals, graph-structured linear systems over the finite field $\mathbb{Z}_p$ don’t have much special structure information which can help speed up the linear system solvers.

2 Reduction

In this section we introduce the main results of this paper: reducing a general linear system over the finite filed $\mathbb{Z}_p$ to a Laplacian linear system over the finite field $\mathbb{Z}_p$, reducing a Laplacian linear system over the finite field $\mathbb{Z}_p$ to a unit weight Laplacian linear system over the finite field $\mathbb{Z}_p$ without adding too many non-zero entries. In Section 2.1 we first introduce some basic definitions and notions. In Section 2.2 we describe the details of reducing a general linear system to a Laplacian linear system over the finite field $\mathbb{Z}_p$. In Section 2.3 we further describe how to reduce a general Laplacian linear system to a unit weight Laplacian linear system over the finite field $\mathbb{Z}_p$.

2.1 Preliminaries

Over the reals, for every simple, weighted, undirected graph $G = (V, E, w)$ with $|V|$ vertices, $|E|$ edges, $w_e > 0$ for all $e \in E$, recall that its corresponding Laplacian matrix $L$
is defined as:

\[
L_{a,b} = \begin{cases} 
\sum_{(a,u) \in E} w_{a,u}, & a = b \\
-w_{a,b}, & (a, b) \in E \\
0, & \text{otherwise}
\end{cases}
\]

Thus Laplacian matrices over the reals are symmetric matrices with all off-diagonal entries negative and 1 in its null space.

We have similar definitions for Laplacian matrices and unit weight Laplacian matrices over the finite field \(\mathbb{Z}_p\).

**Definition 1.** For any integer \(p \geq 2\), a Laplacian matrix over the finite field \(\mathbb{Z}_p\) is a symmetric matrix \(L \in \mathbb{Z}_p^{n \times n}\) with \(L1 \equiv 0 \mod p\), a unit weight Laplacian matrix over the finite field \(\mathbb{Z}_p\) is a Laplacian matrix \(L \in \mathbb{Z}_p^{n \times n}\) with \(L_{ij} \in \{0, p-1\}\) for \(\forall i \in [n], j \in [n], i \neq j\) in addition.

For every Laplacian matrix \(L\) over the finite field \(\mathbb{Z}_p\), we give the definition of its corresponding simple, weighted, undirected graph \(G(L)\):

**Definition 2.** For any integer \(p \geq 2\), a Laplacian matrix \(L \in \mathbb{Z}_p^{n \times n}\), its corresponding simple, weighted, undirected graph is defined as \(G(L) = (V, E, w)\) with \(V = [n], E = \{(u, v) | u > v, L_{u,v} > 0\}, w_{u,v} = p - L_{u,v}, \forall (u, v) \in E\).

In order to characterize over the finite field \(\mathbb{Z}_p\), how close a Laplacian matrix \(L\) is to a unit weight Laplacian matrix, we count the number of edge in \(G(L)\) whose weight is not one:

**Definition 3.** For any integer \(p\), Laplacian matrix \(L \in \mathbb{Z}_p^{n \times n}\), its corresponding simple, weighted, undirected graph \(G(L)\), let \(nnu(L)\) be the number of edges in \(G(L)\) whose weight is not one, i.e. \(|E_1|\) where \(E_1 = \{e | w(e) \neq 1, e \in E(G(L))\}\).

Thus over the finite field \(\mathbb{Z}_p\), a Laplacian matrix \(L\) is a unit weight Laplacian matrix if and only if \(nnu(L) = 0\).

Moreover, similar to the infinite field \(\mathbb{R}\), over the finite field \(\mathbb{Z}_p\), one matrix \(A\) is non-singular if and only if there exists some matrix \(A^{-1}\) such that \(A^{-1}A \equiv AA^{-1} \equiv I \mod p\).

In the remainder part of this paper, we use \(\vec{e}_i\) to denote the \(i\)-th vector in the standard basis.

### 2.2 Reduction to Laplacian Linear Systems

We first perform the reduction from solving general linear systems over \(\mathbb{Z}_p\) to solving Laplacian linear systems over \(\mathbb{Z}_p\). The key observation is the sign constraint for off-diagonal entries no longer exists over the finite field \(\mathbb{Z}_p\). Therefore we only need to come up with a reduction to make the linear system symmetric, have vector of all ones in its nullspace and preserve solutions to the original linear system.
Theorem 2.1. For any prime $p$, $A \in \mathbb{Z}_{p}^{n \times n}$ and $b \in \mathbb{Z}_{p}^{n}$, solving linear system $Ax \equiv b \mod p$ can be reduced to solving a Laplacian system $Ly \equiv c \mod p$ with $\text{nnz}(L) = O(\text{nnz}(A))$.

Proof. Inspired by [KOSZ13], we consider the following linear system
\[
Ly \equiv c \mod p \text{ where } L \equiv \begin{pmatrix}
0 & A & 0 & -A \\
AT & 0 & -AT & 0 \\
0 & -A & 0 & A \\
-AT & 0 & AT & 0
\end{pmatrix},
\]
\[
c \equiv \begin{pmatrix}
b \\
0 \\
-b \\
0
\end{pmatrix}
\]
It's easy to verify $L$ is symmetric and we have
\[
L1 \equiv \begin{pmatrix}
0 & A & 0 & -A \\
AT & 0 & -AT & 0 \\
0 & -A & 0 & A \\
-AT & 0 & AT & 0
\end{pmatrix}\begin{pmatrix}1 \end{pmatrix} \equiv \begin{pmatrix}A - A \\
AT - AT \\
A - A \\
AT - AT
\end{pmatrix} \equiv 0 \mod p
\]
which means $1$ is in the nullspace of $L$. Therefore $L$ is a Laplacian matrix and $\text{nnz}(L) = O(\text{nnz}(A))$.

Considering an solution $y = (y_1, y_2, y_3, y_4)$ to the above linear system, we have
\[
\begin{align*}
Ay_2 - Ay_4 & \equiv b \mod p \\
ATy_1 - ATy_3 & \equiv 0 \mod p \\
-Ay_2 + Ay_4 & \equiv b \mod p \\
-ATy_1 + ATy_3 & \equiv 0 \mod p
\end{align*}
\]
implies $A(y_2 - y_4) \equiv b \mod p$.

Thus $(y_2 - y_4)$ is a solution to the linear system $Ax \equiv b \mod p$.

And for every solution $x$ to $Ax \equiv b \mod p$, we have
\[
\begin{pmatrix}
0 & A & 0 & -A \\
AT & 0 & -AT & 0 \\
0 & -A & 0 & A \\
-AT & 0 & AT & 0
\end{pmatrix}\begin{pmatrix}x \end{pmatrix} \equiv \begin{pmatrix}Ax \\
0 \\
-Ax \\
0
\end{pmatrix} \equiv \begin{pmatrix}b \\
0 \\
-b \\
0
\end{pmatrix} \equiv c \mod p
\]
Thus $y = (0, x, 0, 0)$ is a solution to the Laplacian linear system $Ly \equiv c \mod p$.

Notice this reduction runs in $O(1)$ time, thus over the finite field $\mathbb{Z}_p$, solving a Laplacian linear system is as hard as solving a general linear system.

2.3 Reduction to Unit Weight Laplacian Linear Systems

Secondly we perform the reduction from solving Laplacian linear system over $\mathbb{Z}_p$ to solving unit weight Laplacian linear system over $\mathbb{Z}_p$. The key intuition is that for any linear system $Ax \equiv b \mod p$ over the finite field $\mathbb{Z}_p$, if $A$ is the Schur complement of some block matrix
in \(L\), then we can build a new linear system \(Ly \equiv c \mod p\) which preserves the solution to \(Ax \equiv b \mod p\). Besides, computing Schur complement is in fact doing Gauss elimination. By constructing bigger \(L\) with \(A\) being the Schur complement of some block matrix in \(L\), we are actually doing the Gauss elimination reversely. Therefore the idea behind the reduction from Laplacian linear system to unit weight Laplacian linear system is that every time the current Laplacian matrix \(L\) is not unit weight, we build a larger Laplacian matrix \(\hat{L}\) by eliminating one edge in \(G(L)\) whose weight is non-unit to one unit weight edge, i.e. \(nnu(\hat{L}) = nnu(L) - 1\).

The following lemma shows that Schur complement preserves the solution to linear system.

**Lemma 2.2.** For any \(p, A \in \mathbb{Z}_{p}^{n \times n}\) and \(b \in \mathbb{Z}_{p}^{n}\), solving linear system \(Ax \equiv b \mod p\) can be reduced to solving a linear system \(\hat{A}\hat{x} \equiv \hat{b} \mod p\) with \(\hat{A} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}\) and \(A_4\) is non-singular, \(A \equiv A_1 - A_2A_4^{-1}A_3 \mod p\).

**Proof.** Given a solution \(x\) to the linear system \(Ax \equiv b \mod p\), we have

\[
\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} x \\ -A_4^{-1}A_3x \end{pmatrix} \equiv \begin{pmatrix} Ax \\ 0 \end{pmatrix} \equiv \begin{pmatrix} b \\ 0 \end{pmatrix} \mod p
\]

Thus \(\hat{x} = \begin{pmatrix} x \\ -A_4^{-1}A_3x \end{pmatrix}\) is a solution to the linear system \(\hat{A}\hat{x} \equiv \hat{b} \mod p\).

And for every solution \(\hat{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\) to the linear system \(\hat{A}\hat{x} \equiv \hat{b} \mod p\), we have

\[
\begin{pmatrix} I & -A_2A_4^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} I & -A_2A_4^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} \mod p
\]

\[
\Rightarrow \begin{pmatrix} A_1 - A_2A_4^{-1}A_3 & 0 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} b \\ 0 \end{pmatrix} \mod p
\]

which implies \(Ax_1 \equiv b \mod p\). Therefore \(x_1\) is a solution to the linear system \(Ax \equiv b \mod p\).

In the following lemma, we formally introduce how we eliminate the non-unit weight edges in \(G(L)\) by adding a small amount of non-zero entries to \(L\).

**Lemma 2.3.** For any prime \(p\), Laplacian matrix \(L \in \mathbb{Z}_{p}^{n \times n}\) with \(nnu(L) > 0\) and \(c \in \mathbb{Z}_{p}^{n}\), solving the Laplacian linear system \(Ly \equiv c \mod p\) can be reduced to solving a Laplacian linear system \(\hat{L}\hat{y} \equiv \hat{c} \mod p\) with \(\hat{L} = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}, \hat{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}\), \(L_3\) is non-singular, \(L \equiv L_1 - L_2L_3^{-1}L_2^T \mod p\) and \(nnu(\hat{L}) = nnu(L) - 1, nnz(\hat{L}) = O(nnz(L) + p)\).
Proof. Notice if $p = 2$, by definition of $G(L)$, every edge in $G(L)$ has unit weight, thus $nnu(L)$ is guaranteed to be zero. So in the following proof we assume $p \geq 3$.

First we prove by construction that we can construct a Laplacian matrix $\hat{L} = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ such that $L_3$ is non-singular, $L \equiv L_1 - L_2L_3^{-1}L_2^T \mod p$ and $nnu(\hat{L}) = nnu(L) - 1$, $nnz(\hat{L}) = O(nnz(L) + p)$.

Since $nnu(L) > 0$, $\exists e = uv \in E(G(L))$ such that $w_e > 1$. We construct $L_3$ as:

$$L_3 \equiv 2I_k \mod p$$

where $k = 2(w_e - 1)$.

Notice according to Fermat’s little theorem, $2^{p-1} \equiv 1 \mod p$ for any prime $p \geq 3$. Thus $L_3$ is non-singular and $L_3^{-1} \equiv 2^{p-2}I_k \mod p$. Further we set $L_2 \equiv -(\vec{e}_u + \vec{e}_v)1^T \mod p$, $L_1 \equiv L + L_2L_3^{-1}L_2^T \mod p$. Here $L_2 \in \mathbb{Z}^{n \times k}_p, L_1 \in \mathbb{Z}^{n \times n}_p$.

Then we prove the constructed matrix $\hat{L} = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ is a Laplacian matrix. First as $L$ is a Laplacian matrix and $L_3$ is two times an identity matrix, we have $L = L^T, L_3 = L_3^T$.

Thus

$$\hat{L}^T \equiv \begin{pmatrix} L_1^T & L_2 \\ L_2^T & L_3 \end{pmatrix} \equiv \begin{pmatrix} (L + L_2L_3^{-1}L_2^T) & L_2 \\ L_2^T & L_3 \end{pmatrix} \equiv \begin{pmatrix} L + L_2L_3^{-1}L_2^T & L_2 \\ L_2^T & L_3 \end{pmatrix} \equiv \hat{L} \mod p$$

which shows $\hat{L}$ is symmetric. Besides we have

$$(L_2^T \ L_3) \ 1 \equiv (-1(\vec{e}_u + \vec{e}_v)^T \ 2I_k) \ 1 \equiv 0 \mod p$$

Then because $L$ is a Laplacian matrix, $L \equiv 0 \mod p$, we have

$$\begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix} \ 1 \equiv \begin{pmatrix} L + L_2L_3^{-1}L_2^T & L_2 \\ L_2^T & L_3 \end{pmatrix} \ 1$$

$$\equiv L_1 + L_2L_3^{-1}(L_2^T + L_3) \ 1$$

$$\equiv L_2L_3^{-1} \ 0$$

$$\equiv 0 \mod p$$

Thus

$$\hat{L} \ 1 \equiv \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix} \ 1 \equiv 0 \mod p$$

Therefore $\hat{L} = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ is a Laplacian matrix.

To show $nnu(\hat{L}) = nnu(L) - 1$, we first show $\forall e = ij \in G(L) - uv, w_{G(L)}(e) = w_{G(\hat{L})}(e)$. This is because by definition:

$$w_{G(\hat{L})}(e) \equiv (p - \hat{L}_{i,j}) \equiv p - (L_{i,j})$$

$$\equiv p - \vec{e}_i^T(\vec{e}_u + \vec{e}_v)1^T(L_2^T + L_3) \vec{e}_j$$

$$\equiv (p - L_{i,j}) + \vec{e}_i^T((\vec{e}_u + \vec{e}_v)1^T(L_2^T + L_3) \vec{e}_j$$

$$\equiv w_{G(L)}(e) \mod p$$
The last step is due to $ij \neq uv$. And we have

$$w_{G(L)}(uv) \equiv (p - \hat{L}_{u,v}) \equiv p - (L_1)_{u,v}$$
$$\equiv p - \bar{e}_u^T(L + L_2L_3^{-1}L_2^T)\bar{e}_v$$
$$\equiv (p - L_{u,v}) + \bar{e}_u^T((\bar{e}_u + \bar{e}_v)1^TL_3^{-1}1(\bar{e}_u + \bar{e}_v)^T)\bar{e}_v$$
$$\equiv (p - L_{u,v}) + 1^TL_3^{-1}1$$
$$\equiv (p - L_{u,v}) + 2^{p-2} \cdot 2(p - 1 - L_{u,v})$$
$$\equiv 1 \mod p$$

And by the definition of $L_2$ and $L_3$, we know the newly added edges in $G(\hat{L})$ are all unit weight edges, therefore we have $nnu(\hat{L}) = nnu(L) - 1$.

Based on the conclusion that there are only one edge in $G(L)$ whose weight is changed, we have $nnz(L_1) = O(nnz(L))$. By the definition of $L_2$ and $L_3$, we have $nnz(L_2) = O(k) = O(p)$, $nnz(L_3) = O(k) = O(p)$. Thus $nnz(\hat{L}) = nnz(L_1) + 2nnz(L_2) + nnz(L_3) = O(nnz(L) + p)$.

Thus we constructed a Laplacian matrix $\hat{L}$ with $\hat{L} = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$, $L_3$ is non-singular, $L \equiv L_1 - L_2L_3^{-1}L_2^T \mod p$ and $nnu(\hat{L}) = nnu(L) - 1$, $nnz(\hat{L}) = O(nnz(L) + p)$, by Lemma 2.2 we know solving linear system $L\hat{y} \equiv \hat{c} \mod p$ can be reduced to solving linear system $\hat{L}\hat{y} \equiv \hat{c} \mod p$ where $\hat{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}$.

The above lemma shows we can eliminate one non-unit weight edge by adding $O(p)$ non-zero entries. Notice the time complexity of construct $\hat{L}$ is $O(p)$ since computing $L_1, L_2, L_3$ all costs $O(p)$ time. Using the above lemma, we can prove the following theorem:

**Theorem 2.4.** For any prime $p$, Laplacian matrix $L \in \mathbb{Z}_p^{n \times n}$ and $c \in \mathbb{Z}_p^n$, solving a Laplacian linear system $Lx \equiv c \mod p$ can be reduced to solving a unit weight Laplacian system $\hat{L}\hat{y} \equiv \hat{c} \mod p$ with $nnz(\hat{L}) = O(p \cdot nnz(L))$.

**Proof.** By Lemma 2.3 we can eliminate one non-unit weight edge in $G(L)$ by adding $O(p)$ non-zero entries. Therefore by applying the construction in Lemma 2.3 $nnu(L)$ times, we can reduce the Laplacian linear system $L\hat{y} \equiv c \mod p$ to a unit weight Laplacian linear system $\hat{L}\hat{y} \equiv \hat{c} \mod p$ with $nnz(\hat{L}) = O(nnz(L) + p \cdot nnu(L)) = O(p \cdot nnz(L))$. \qed
**Algorithm 1** Reduce solving a Laplacian linear system $Ly = c \mod p$ to solving a unit weight Laplacian system $\hat{L}\hat{y} = \hat{c} \mod p$

1: **Input:** $n$, Laplacian matrix $L \in \mathbb{Z}^{n \times n}_p$, $c \in \mathbb{Z}^n_p$, prime $p$
2: **procedure** MODIFY_EDGE_WEIGHT($L, c, u, v, p$) $\triangleright$ Make the edge between $u$ and $v$ unit weight
3: $m \leftarrow \text{size}(L)$ $\triangleright$ size of current $L$
4: $k \leftarrow 2(p - 1 - L_{uv})$ $\triangleright$ $L_2 \in \mathbb{Z}^{m \times k}_p$
5: $L_2 \equiv -(\vec{e}_u + \vec{e}_v)1^T \mod p$
6: $L_3 \equiv 2I_k \mod p$ $\triangleright$ $L_3 \in \mathbb{Z}^{k \times k}_p$
7: $L_1 \equiv L + L_2L_3^{-1}L_2^T \mod p$ $\triangleright$ $\hat{L} \in \mathbb{Z}_p^{(m+k)\times(m+k)}$
8: $\hat{L} \leftarrow \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$
9: $\hat{c} \leftarrow \begin{pmatrix} c \\ 0 \end{pmatrix}$ $\triangleright$ $\hat{c} \in \mathbb{Z}_p^{(m+k)}$
10: **Return:** $\hat{L}, \hat{c}$
11: **end procedure**
12: **procedure** REDUCTION($n, L, c, p$) $\triangleleft$ Reduce $Ly = c \mod p$ to $\hat{L}\hat{y} = \hat{c} \mod p$ where $\hat{L}$ is a unit weight Laplacian matrix
13: $\hat{L} \leftarrow L, \hat{c} \leftarrow c$
14: for $i \in [n]$ do
15: for $j \in [i - 1]$ do
16: if $L_{ij} \notin \{0, p - 1\}$ then
17: $\hat{L}, \hat{c} \leftarrow \text{ELIMINATE}(\hat{L}, \hat{c}, i, j, p)$
18: end if
19: end for
20: end for
21: **end procedure**

**References**


