
Reading: *Principles of Robot Motion* Chapter 3 and Appendix E. In addition, you may find it helpful to consult the suggested readings that are listed in the lecture schedule for the course.

Configuration space: topology, parameterizations

Problems:

1. Determine the configuration space for each of the following.

(a) A mobile robot that can translate and rotate in the plane.
Soln: $\mathcal{Q} = SE(2)$

(b) A six-link anthropomorphic arm.
Soln: $\mathcal{Q} = \mathcal{T}^6$, the 6-torus

(c) A quadrotor.
Soln: $\mathcal{Q} = SE(3)$, since the quadrotor is essentially a rigid object moving in $\Re^3$.

(d) A mobile manipulator that comprises a robot base (which can rotate and translate in the plane) and a six-link anthropomorphic arm.
Soln: $\mathcal{Q} = SE(2) \times \mathcal{T}^6$, which is the direct product of the configuration spaces for the mobile manipulator and the arm.

(e) A simple bipedal robot with two legs and a torso, each leg attached to the torso by one revolute joint, and each leg containing one revolute knee joint.
Soln: $\mathcal{Q} = SE(3) \times \mathcal{T}^4$. We can take any link of the robot as the reference link, and since this link is a rigid body, its configuration space is $SE(3)$. Once we fix the position of one link, this robot can be viewed as a kinematic chain with four joints, whose configuration space is the 4-torus.

2. Construct an atlas for $SO(2)$ consisting of two charts. You must show that the two charts satisfy the conditions required for an atlas.
Soln: To construct an atlas of two charts, we use the parameterization

$$R = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Define $U_1 = \{R \in SO(2) \mid x_1 \neq 1\}$, and $U_2 = \{R \in SO(2) \mid x_1 \neq -1\}$, i.e., U_1 is the set of all rotation matrices such that $\theta \neq 0$, i.e., U_2 is the set of all rotation matrices such that $\theta \neq \pi$.

The two argument arctangent function Atan2 : $\Re^2 \backslash (0,0) \rightarrow [-\pi, \pi]$ maps the point x, y in the plane to the unique angle $\mathrm{Atan2}(y, x) = \theta \in [-\pi, \pi]$ such that $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$, and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$.

Using Atan2, we may define ϕ_i as

$$\phi_1(R) = \begin{cases} \mathrm{Atan2}(x_3, x_1), & x_3 \geq 0 \\ \mathrm{Atan2}(x_3, x_1) + 2\pi, & x_3 < 0 \end{cases}$$

$$\phi_2(R) = \mathrm{Atan2}(x_3, x_1)$$

The intersection $U_1 \cap U_2$ consists of two disjoint subsets: $V_1 = \{R \in SO(2) \mid x_3 > 1\}$ and $V_2 = \{R \in SO(2) \mid x_3 < 1, \}$. To show that the two charts are C^∞-related, we merely compute $\phi_1 \circ \phi_2^{-1}$ and $\phi_2 \circ \phi_1^{-1}$ for V_1 and V_2. For $R \in V_1$ we have $\phi_1 \circ \phi_2^{-1}(\theta) = \theta$, which is (obviously) smooth. For $R \in V_2$ we have $\phi_1 \circ \phi_2^{-1}(\theta) = \theta + 2\pi$, which is also smooth. In the other direction $\phi_2 \circ \phi_1^{-1} = \theta$ for both V_1 and V_2.

3. Consider the ZYX Euler angles α, β, γ such that $R = R_{z,\alpha}R_{y,\beta}R_{x,\gamma}$ and $R \in SO(3)$. Show that these Euler angles cannot be used to construct a global chart for $SO(3)$. What is the relationship between these Euler angles and roll, pitch and yaw angles?
Soln: The equation for $R = R_{z,\alpha}R_{y,\beta}R_{x,\gamma}$ is given in eq. E.12 of [Choset]. Thus α, β, and γ are the roll, pitch and yaw angles (respectively). For $\theta = \pm\pi/2$, the matrix of eq. E.12 takes the form

$$R = \begin{bmatrix} 0 & -s_\phi c_\psi + c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi \\ 0 & s_\phi s_\psi + c_\phi c_\psi & -c_\phi s_\psi + s_\phi c_\psi \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & s_{\psi-\phi} & c_{\psi-\phi} \\ 0 & c_{\psi-\phi} & -s_{\psi-\phi} \\ -1 & 0 & 0 \end{bmatrix}$$

and in this case there are infinitely many solutions for ψ and ϕ. This is analogous to the Z-Y-Z Euler angle singularity for $\theta = 0$.

4. The two dimensional torus $\mathcal{T}^2$ embedded in $\Re^3$ can be defined by

$$f : \Re^2 \to \Re^3, \qquad f(\theta_1, \theta_2) = ((R + r\cos\theta_1)\cos\theta_2, (R + r\cos\theta_1)\sin\theta_2, r\sin\theta_1)$$

in which R is called the major radius and r is the minor radius. Since f is not a bijection, it cannot be used to define a single global chart on $\mathcal{T}^2$. However, it is easy to define a charts on the torus by using f and restricting its domain. For example, let

$$\begin{aligned} V_1 &= \{(\theta_1, \theta_2) \in \Re^2 \mid 0 < \theta_1 < 2\pi,\ 0 < \theta_2 < 2\pi\} \\ V_2 &= \{(\theta_1, \theta_2) \in \Re^2 \mid 0 < \theta_1 < 2\pi,\ -\pi < \theta_2 < \pi\} \end{aligned}$$

and let U_i denote the image of V_i under f, i.e.,

$$f(V_i) = \{(x, y, z) \in \Re^3 \mid (\theta_i, \theta_j) \in V_i,\ \ f(\theta_1, \theta_2) = (x, y, z)\} = U_i \subset \mathcal{T}^2$$

Then we can define the charts (U_i, ϕ_i), with $\phi_i : U_i \to V_i$ defined by $\phi_i = f^{-1}(x, y, z)$.

(a) Sketch the sets U_1 and U_2 (draw two separate tori).

(b) Show that the charts (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞ related.
Soln: The compositions $\phi_1 \circ \phi_2^{-1}$ and $\phi_2 \circ \phi_1^{-1}$ are merely piecewise translations (as in Problem 2 above), and are thus smooth.

(c) Construct an atlas for $\mathcal{T}^2$ using (U_1, ϕ_1) and (U_2, ϕ_2), and defining as many additional (U_i, ϕ_i) as necessary. ***You do not need to show that the collection of charts is C^∞ related;*** you demonstrated your ability to do so in Part b.
Soln: We need two additional charts, which can be defined in a manner analogous to the definition of U_1 and U_2:

$$\begin{aligned} V_3 &= \{(\theta_1, \theta_2) \in \Re^2 \mid -\pi < \theta_1 < \pi,\ 0 < \theta_2 < 2\pi\} \\ V_4 &= \{(\theta_1, \theta_2) \in \Re^2 \mid -\pi < \theta_1 < \pi,\ -\pi < \theta_2 < \pi\} \end{aligned}$$

and let U_i denote the image of V_i under f given above.

5. The torus $\mathcal{T}^2$ can also be defined by the constraint $\left(R - \sqrt{x^2+y^2}\right)^2 + z^2 - r^2 = 0$, i.e.,

$$\mathcal{T}^2 = \{(x,y,z) \in \Re^3 \mid \left(R - \sqrt{x^2+y^2}\right)^2 + z^2 - r^2 = 0\}$$

Use the implicit function to show that the torus is a manifold of dimension 2. Note: you will need to apply the implicit theorem more than once.

Soln: First, consider the variable assignment $u_1 = x$, $u_2 = y$, and $v = z$, and write the constraint as

$$f(u_1, u_2, v) = \left(R - \sqrt{u_1^2 + u_2^2}\right)^2 + v^2 - r^2 = 0$$

(note, since we use r to denote the minor radius, we use u and v as the variables for application of the implicit function theorem). The Jacobian of f w.r.t. v is given by $J = 2v$, and thus for $v = z \neq 0$ we can parameterize the torus by $u_1 = x, u_2 = y$.

Now consider the variable assignment $u_1 = x$, $v = y$, and $u_2 = z$, and write the constraint as

$$f(u_1, u_2, v) = \left(R - \sqrt{u_1^2 + v^2}\right)^2 + u_2^2 - r^2 = 0$$

and the Jacobian is given by

$$J = \frac{\partial}{\partial v} f = -2\left(R - \sqrt{u_1^2+v^2}\right)\left(\frac{v}{\sqrt{u_1^2+v^2}}\right)$$

In this case, we have $J = 0$ when $v = y = 0$ or when $R - \sqrt{u_1^2+v^2} = R - \sqrt{x^2+y^2} = 0$. In the latter case, we have $z = \pm r$. Thus, for $z \neq \pm r$ and $y \neq 0$, we can parameterize the torus by $u_1 = x, u_2 = z$. At this point, we have covered the entire torus except for two points, $x = R \pm r, y = 0, z = 0$.

Finally, consider the variable assignment $v = x$, $u_1 = y$, and $u_2 = z$, and write the constraint as

$$f(u_1, u_2, v) = \left(R - \sqrt{v^2 + u_1^2}\right)^2 + u_2^2 - r^2 = 0$$

and the Jacobian is given by

$$J = \frac{\partial}{\partial v} f = -2\left(R - \sqrt{v^2+u_1^2}\right)\left(\frac{v}{\sqrt{v^2+u_1^2}}\right)$$

In this case, we have $J = 0$ when $v = y = 0$ or when $R - \sqrt{v^2+u_1^2} = R - \sqrt{x^2+y^2} = 0$. In the latter case, we have $z = \pm r$. Thus, for $z \neq \pm r$ and $z \neq 0$, we can parameterize the torus by $u_1 = y, u_2 = z$.

Since the three cases above cover the entire torus, we may conclude that the torus is a 2-manifold.

6. For a unit quaternion Q, let $R(Q)$ denote the corresponding rotation matrix (see eqn. E.28 of the text).

(a) For $v = (v_1, v_2, v_3)$, show that $v' = R(Q)v$ is given by $(0, v') = Q(0, v)Q^*$. There are several possible solutions; the most straightforward is to work out the Quaternion product, and show that the result is equal to the product $R(Q)v$. Some hints: The vector triple product and the scalar triple product might be useful. The vector cross product operation is neither commutative nor associative.

Soln: Let's simply work out the Quaternion product:

$$\begin{aligned}
Q(0,v)Q^* &= (q_0, q)(0, v)(q_0, -q) \\
&= (-q^T v, q_0 v + q \times v)(q_0, -q)
\end{aligned}$$

Let's consider the scalar and vector parts individually. For the scalar part we have

$$\begin{aligned}
\text{Scalar part} &= (-q^T v)q_0 - (q_0 v + q \times v)^T(-q) \\
&= (-q^T v)q_0 + q_0 v^T q + (q \times v)^T q \\
&= (q \times v)^T q \\
&= q \cdot (q \times v) \qquad \text{Now, apply the scalar triple product: } a \cdot (b \times c) = c \cdot (b \times a) \\
&= v \cdot (q \times q) = 0
\end{aligned}$$

Thus, the scalar part satisfies the required condition.

Now, for the vector part we have a bit of tedious vector algebra:

$$\begin{aligned}
\text{Vector part} = v' &= (-q^T v)(-q) + q_0(q_0 v + q \times v) + (q_0 v + q \times v) \times (-q) \\
&= (q^T v)q + q_0^2 v + q_0(q \times v) - (q_0 v + q \times v) \times q \\
&= (q^T v)q + q_0^2 v + q_0(q \times v) + q \times (q_0 v + q \times v) \quad \text{since } a \times b = -b \times a \\
&= (q^T v)q + q_0^2 v + q_0(q \times v) + q_0(q \times v) + q \times (q \times v) \\
&\qquad \textit{Now use vector triple product:} \quad a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \\
&= (q^T v)q + q_0^2 v + q_0(q \times v) + q_0(q \times v) + q(q^T v) - v(q^T q) \\
&= 2(q^T v)q + q_0^2 v + 2q_0(q \times v) - (q^T q)v
\end{aligned}$$

Now, simply carry out the multiplications, using the notation $q = (q_1, q_2, q_3)^T$ and $v = v_1, v_2, v_3)^T$:

$$\begin{aligned}
&= 2(q_1 v_1 + q_2 v_2 + q_3 v_3)\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + q_0^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + 2q_0 \begin{bmatrix} -q_3 v_2 + q_2 v_3 \\ q_3 v_1 - q_1 v_3 \\ -q_2 v_1 + q_1 v_2 \end{bmatrix} - (q_1^2 + q_2^2 + q_3^2)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
&= \begin{bmatrix} 2q_1^2 v_1 + 2q_1 q_2 v_2 + 2q_1 q_3 v_3 + q_0^2 v_1 - 2q_0 q_3 v_2 + 2q_0 q_2 v_3 - q_1^2 v_1 - q_2^2 v_1 - q_3^2 v_1 \\ 2q_1 q_2 v_1 + 2q_2^2 v_2 + 2q_2 q_3 v_3 + q_0^2 v_2 + 2q_0 q_3 v_1 - 2q_0 q_1 v_3 - q_1^2 v_2 - q_2^2 v_2 - q_3^2 v_2 \\ 2q_1 q_3 v_1 + 2q_2 q_3 v_2 + 2q_3^2 v_3 + q_0^2 v_3 - 2q_0 q_2 v_1 + 2q_0 q_1 v_2 - q_1^2 v_3 - q_2^2 v_3 - q_3^2 v_3 \end{bmatrix} \\
&= \begin{bmatrix} 2q_1^2 + q_0^2 - q_1^2 - q_2^2 - q_3^2 & +2q_1 q_2 - 2q_0 q_3 & +2q_1 q_3 + 2q_0 q_2 \\ 2q_1 q_2 + 2q_0 q_3 & 2q_2^2 + q_0^2 - q_1^2 - q_2^2 - q_3^2 & 2q_2 q_3 - 2q_0 q_1 \\ 2q_1 q_3 - 2q_0 q_2 & 2q_2 q_3 + 2q_0 q_1 & q_3^2 + q_0^2 - q_1^2 - q_2^2 - q_3^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
&\qquad \text{Since we are dealing with unit quaternions: } -q_1^2 - q_2^2 - q_3^2 = q_0^2 - 1 \\
&= \begin{bmatrix} 2(q_o^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}
\end{aligned}$$

And this expression matches exactly eq. E.28 of [Choset].

(b) For unit quaternions Q_1 and Q_2, show that the composite rotation is given by $Q = Q_1Q_2$, i.e., show that $R(Q) = R(Q_1)R(Q_2)$ for $Q = Q_1Q_2$. Hint: You should not need eqn. E.28 for this demonstration.

Soln: Consider a vector v, and define v' and v'' as

$$\begin{aligned} v' &= R(Q_2)v \\ v'' &= R(Q_1)v' = R(Q_1)R(Q_2)v \end{aligned}$$

From Part a, we have

$$\begin{aligned} (0, v') &= Q_2(0, v)Q_2^* \\ (0, v'') &= Q_1(0, v')Q_1^* \end{aligned}$$

Substituting the first equation into the second yields

$$\begin{aligned} (0, v'') &= Q_1 \left(Q_2(0, v)Q_2^*\right) Q_1^* \\ &= (Q_1Q_2)\,(0, v)\,(Q_2^*Q_1^*) \end{aligned}$$

(c) For unit quaternions Q_1 and Q_2, show that $(Q_1Q_2)^* = Q_2^*Q_1^*$.
Soln: It suffices to show that $(Q_1Q_2)(Q_2^*Q_1^*) = (Q_2^*Q_1^*)(Q_1Q_2) = (1, 0)$, where $(1, 0)$ is the identity element of $\mathcal{Q}$, which corresponds to no rotation. Using the associativity of quaternion multiplication, and the fact that $QQ^* = Q^*Q = (1, 0)$ (see eq. E.32), we have

$$\begin{aligned} (Q_1Q_2)(Q_2^*Q_1^*) &= Q_1(Q_2Q_2^*)Q_1^* = Q_1Q_1^* = (1, 0) \\ (Q_2^*Q_1^*)(Q_1Q_2) &= Q_2^*(Q_1^*Q_1)Q_2 = Q_2^*Q_2 = (1, 0) \end{aligned}$$

(d) Show that $R^T(Q) = R(Q^*)$.
Soln:

$$\begin{aligned} I &= R(QQ^*) \quad \text{since } QQ^* \text{ corresponds to the null rotation} \\ &= R(Q)R(Q^*) \quad \text{using the result from part b} \\ R^T(Q) &= R(Q^*) \quad \text{since } R^T(Q) = R^{-1}(Q) \end{aligned}$$

7. Let $\mathcal{Q}$ denote the set of unit quaternions.

(a) Show that $\mathcal{Q}$ is a 3-manifold.

Soln: Unit quaternions satisfy the constraint $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, i.e., the set of unit quaternions is in fact the surface of the unit 3-sphere, i.e., $\mathcal{Q} = S^3$. This suffices to demonstrate that $\mathcal{Q}$ is a 3-manifold, since S^3 is a 3-manifold. To prove the result directly, simply apply the implicit function theorem, using the constraint $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

(b) Show that there does not exist a global diffeomorphism ϕ between $\mathcal{Q}$ and $SO(3)$, i.e., show that there does not exist $\phi : \mathcal{Q} \to SO(3)$, such that ϕ is a C^∞ bijection.

Soln: For any quaternion $Q = (q_0, q)$, the quaternion $Q' = (-q_0, -q)$ satisfies $R(Q) = R(Q')$. This can be seen from eqn. E.28, or by noting that rotation by θ about an axis k is equivalent to rotation by $-\theta$ about the axis $-k$.

Thus, for any $\phi : \mathcal{Q} \to SO(3)$, the inverse mapping ϕ^{-1} is not well defined, i.e., for any R, we have $\phi^{-1}(R) = \{Q, Q'\}$ for $R = \phi(Q)$.

(c) Construct a chart for $\mathcal{Q}$. Since no global chart exists, you must specify both an open set $U \subset \mathcal{Q}$, and a mapping ϕ. You are not required here to construct a full atlas.

Soln: This can be done using the implicit function theorem. The set of unit quaternions $\mathcal{Q}$ is implicitly defined by the constraint $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. To apply the implicit function theorem, rewrite this constraint as $r_1^2 + r_2^2 + r_3^2 + s^2 - 1 = 0$. The Jacobian of this constraint is given by $J = 2s$, and we have $\det J \neq 0$ for $s \neq 0$. Thus, we may define a chart (U, ϕ) by

$U = \{Q = (q_0, q_1, q_2, q_3) \in \mathcal{Q} \mid q_3 \neq 0\}$

$\phi(q_0, q_1, q_2, q_3) = (q_0, q_1, q_2)$ and $\phi^{-1}(q_0, q_1, q_2) = (q_0, q_1, q_2, \sqrt{1 - q_0^2 + q_1^2 + q_2^2})$

8. Many path planning methods require the ability to compute a path in configuration space that connects two distinct configurations. This can be accomplished by interpolating between the two configurations. Here, we consider the problem of interpolating between orientations for several different parameterizations of $SO(3)$ by defining a continuous function g, such that $g(0)$ is the initial orientation and $g(1)$ is the final orientation.

(a) Define a continuous function $g : [0, 1] \to SO(3)$ such that $g(0) = I$ and $g(1) = R$, for a given $R \in SO(3)$. It may be tempting to use a simple linear interpolation of the form $g(t) = I + t(R - I)$. Although this choice of g satisfies the boundary conditions, it is easy to show that $g(t) \notin SO(3)$ for general $t \in (0, 1)$. Find an appropriate g. (Hint, think of axis-angle parameterization).
Soln: Let $R_{k,\theta}$ denote the rotation matrix that corresponds to a rotation by angle θ about the unit vector k (see Appendix E.3 of [Choset]). Then we may define $g(t) = R_{k,\theta t}$. Clearly $R_{k,\theta t} \in SO(3)$ for all t, since θt is merely an angle such that $0 \leq \theta t \leq \theta$ for $t \in [0, 1]$.

(b) Define a continuous function $g : [0, 1] \to SO(3)$ such that $g(0) = R_1$ and $g(1) = R_2$, for given $R_1, R_2 \in SO(3)$.
Soln: Consider a rotation matrix $R(t)$ such that $R_1 R(0) = R_1$ and $R_1 R(1) = R_2$. It is easy to see that $R(0) = I$. Further, if we multiply both sides of $R_1 R(1) = R_2$ by R_1^T we obtain $R(1) = R_1^T R_2$. Thus, we arrive to the problem of finding a rotation matrix $R(t)$ such that $R(0) = I$ and $R(1) = R_1^T R_2$, which is exactly the problem that we solved in part (a) above, i.e., merely define angle θ and axis k such that $R(1) = R_{k,\theta}$, and we arrive to $g(t) = R_1 R_{k,\theta t}$.

(c) For ZYX Euler angles α, β, γ such that $R = R_{z,\alpha} R_{y,\beta} R_{x,\gamma}$, define a continuous function $g : [0, 1] \to SO(3)$ such that $g(0) = I$ and $g(1) = R$, for a given α, β, γ.
Soln: We may extend the results of part (a) above in a straightforward way to obtain the interpolating function $g(t) = R_{z,\alpha t} R_{y,\beta t} R_{z,\gamma t}$. Since $R_{k,\theta t} \in SO(3)$ for all $t \in [0, 1]$ and any axis k (as above), we have $R_{z,\alpha t}, R_{y,\beta t}, R_{z,\gamma} \in SO(3)$ for all $t \in [0, 1]$, and clearly $g(0) = I$ and $g(1) = R_{z,\alpha} R_{y,\beta} R_{x,\gamma}$.

(d) Define a continuous function $g : [0, 1] \to \mathcal{Q}$ such that $g(0) = (1, 0, 0, 0)$ and $g(1) = Q = (q_0, q_1, q_2, q_3)$, for a given $Q \in \mathcal{Q}$.
Soln: From eq. E.26 [Choset] we can express Q as a rotation about an axis,

$$Q = (\cos\theta/2, n_x \sin\theta/2, n_y \sin\theta/2, n_z \sin\theta/2)$$

Thus, we may define $g(t)$ as

$$g(t) = (\cos\theta t/2, n_x \sin\theta t/2, n_y \sin\theta t/2, n_z \sin\theta t/2)$$

and it is clear that $g(0) = (1, 0, 0, 0)$, $g(1) = Q$, and $g(t) \in \mathcal{Q}$ for all $t \in [0, 1]$.