Reading: *Principles of Robot Motion* Chapter 3 and Appendix E. In addition, you may find it helpful to consult the suggested readings that are listed in the lecture schedule for the course.

Configuration space: topology, parameterizations

Problems:

1. Determine the configuration space for each of the following.

(a) A mobile robot that can translate and rotate in the plane.

(b) A six-link anthropomorphic arm.

(c) A quadrotor.

(d) A mobile manipulator that comprises a robot base (which can rotate and translate in the plane) and a six-link anthropomorphic arm.

(e) A simple bipedal robot with two legs and a torso, each leg attached to the torso by one revolute joint, and each leg containing one revolute knee joint.

2. Construct an atlas for $SO(2)$ consisting of two charts. You must show that the two charts satisfy the conditions required for an atlas.

3. Consider the ZYX Euler angles α, β, γ such that $R = R_{z,\alpha}R_{y,\beta}R_{x,\gamma}$ and $R \in SO(3)$. Show that these Euler angles cannot be used to construct a global chart for $SO(3)$.
As a bonus: What is the relationship between these Euler angles and roll, pitch and yaw angles?

4. The two dimensional torus $\mathcal{T}^2$ embedded in $\Re^3$ can be defined by

$$f : \Re^2 \to \Re^3, \qquad f(\theta_1, \theta_2) = ((R + r\cos\theta_1)\cos\theta_2, (R + r\cos\theta_1)\sin\theta_2, r\sin\theta_1)$$

in which R is called the major radius and r is the minor radius. Since f is not a bijection, it cannot be used to define a single global chart on $\mathcal{T}^2$. However, it is easy to define a charts on the torus by using$f$ and restricting its domain. For example, let

$$\begin{aligned} V_1 &= \{(\theta_1, \theta_2) \in \Re^2 \mid 0 < \theta_1 < 2\pi,\ 0 < \theta_2 < 2\pi\} \\ V_2 &= \{(\theta_1, \theta_2) \in \Re^2 \mid 0 < \theta_1 < 2\pi,\ -\pi < \theta_2 < \pi\} \end{aligned}$$

and let U_i denote the image of V_i under f, i.e.,

$$f(V_i) = \{(x, y, z) \in \Re^3 \mid (\theta_i, \theta_j) \in V_i,\ \ f(\theta_1, \theta_2) = (x, y, z)\} = U_i \subset \mathcal{T}^2$$

Then we can define the charts (U_i, ϕ_i), with $\phi_i : U_i \to V_i$ defined by $\phi_i = f^{-1}(x, y, z)$.

(a) Sketch the sets U_1 and U_2 (draw two separate tori).

(b) Show that the charts (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞ related.

(c) Construct an atlas for $\mathcal{T}^2$ using (U_1, ϕ_1) and (U_2, ϕ_2), and defining as many additional (U_i, ϕ_i) as necessary. ***You do not need to show that the collection of charts is C^∞ related;*** you demonstrated your ability to do so in Part b.

5. The torus $\mathcal{T}^2$ can also be defined by the constraint $\left(R - \sqrt{x^2+y^2}\right)^2 + z^2 - r^2 = 0$, i.e.,

$$\mathcal{T}^2 = \{(x,y,z) \in \Re^3 \mid \left(R - \sqrt{x^2+y^2}\right)^2 + z^2 - r^2 = 0\}$$

Use the implicit function to show that the torus is a manifold of dimension 2. Note: you will need to apply the implicit theorem more than once.

6. For a unit quaternion Q, let $R(Q)$ denote the corresponding rotation matrix (see eqn. E.28 of the text).

(a) For $v = (v_1, v_2, v_3)$, show that $v' = R(Q)v$ is given by $(0, v') = Q(0, v)Q^*$. There are several possible solutions; the most straightforward is to work out the Quaternion product, and show that the result is equal to the product $R(Q)v$. Some hints: The vector triple product and the scalar triple product might be useful. The vector cross product operation is neither commutative nor associative.

(b) For unit quaternions Q_1 and Q_2, show that the composite rotation is given by $Q = Q_1Q_2$, i.e., show that $R(Q) = R(Q_1)R(Q_2)$ for $Q = Q_1Q_2$. Hint: You should not need eqn. E.28 for this demonstration.

(c) For unit quaternions Q_1 and Q_2, show that $(Q_1Q_2)^* = Q_2^*Q_1^*$

(d) Show that $R^T(Q) = R(Q^*)$.

7. Let $\mathcal{Q}$ denote the set of unit quaternions.

(a) Show that $\mathcal{Q}$ is a 3-manifold.

(b) Show that there does not exist a global diffeomorphism ϕ between $\mathcal{Q}$ and $SO(3)$, i.e., show that there does not exist $\phi : \mathcal{Q} \to SO(3)$, such that ϕ is a C^∞ bijection.

(c) Construct a chart for $\mathcal{Q}$. Since no global chart exists, you must specify both an open set $U \subset \mathcal{Q}$, and a mapping ϕ. You are not required here to construct a full atlas.

8. Many path planning methods require the ability to compute a path in configuration space that connects two distinct configurations. This can be accomplished by interpolating between the two configurations. Here, we consider the problem of interpolating between orientations for several different parameterizations of $SO(3)$ by defining a continuous function g, such that $g(0)$ is the initial orientation and $g(1)$ is the final orientation.

(a) Define a continuous function $g : [0,1] \to SO(3)$ such that $g(0) = I$ and $g(1) = R$, for a given $R \in SO(3)$. It may be tempting to use a simple linear interpolation of the form $g(t) = I + t(R - I)$. Although this choice of g satisfies the boundary conditions, it is easy to show that $g(t) \notin SO(3)$ for general $t \in (0,1)$. Find an appropriate g. (Hint, think of axis-angle parameterization).

(b) Define a continuous function $g : [0,1] \to SO(3)$ such that $g(0) = R_1$ and $g(1) = R_2$, for given $R_1, R_2 \in SO(3)$.

(c) For ZYX Euler angles α, β, γ such that $R = R_{z,\alpha}R_{y,\beta}R_{x,\gamma}$, define a continuous function $g : [0,1] \to SO(3)$ such that $g(0) = I$ and $g(1) = R$, for a given α, β, γ.

(d) Define a continuous function $g : [0,1] \to \mathcal{Q}$ such that $g(0) = (1,0,0,0)$ and $g(1) = Q = (q_0, q_1, q_2, q_3)$, for a given $Q \in \mathcal{Q}$.