The Price of Fair PCA: One Extra Dimension

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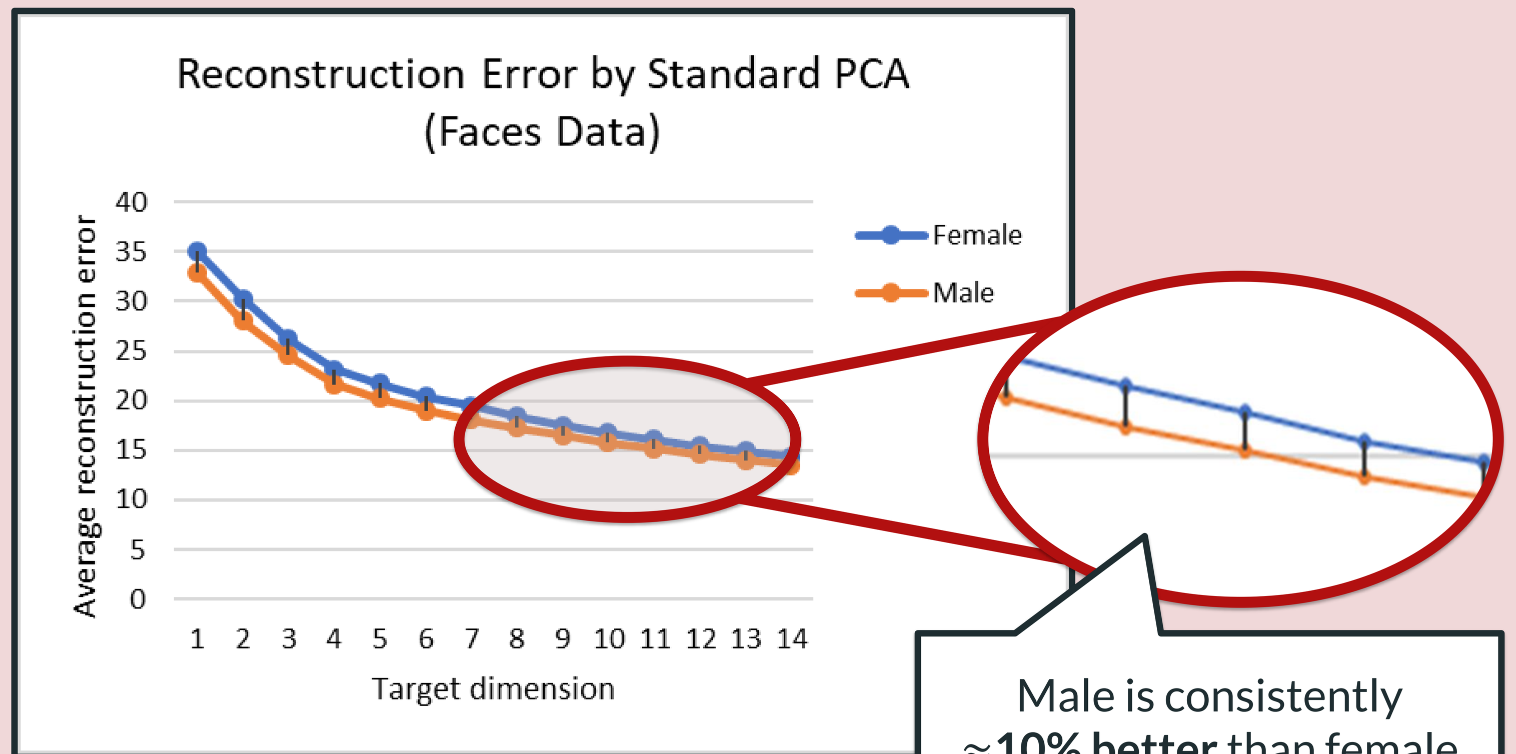
Motivation

- Principle Component Analysis (PCA) is used in machine learning, natural sciences, and social sciences
- PCA can result in **biased representation**



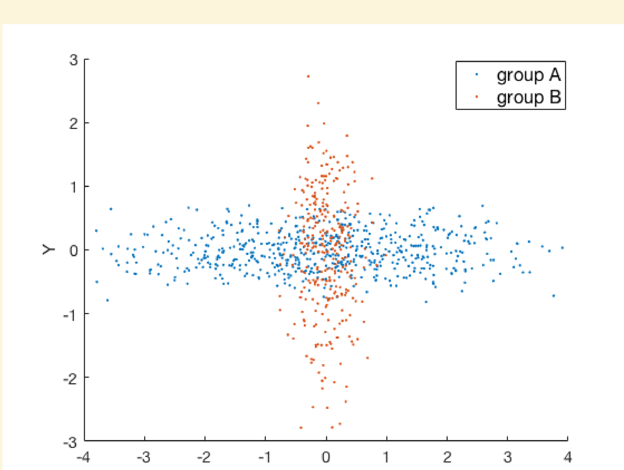
- Reweighting samples from each group to be equal does not fix the bias
- Two PCAs for each group are not allowed for ethical and legal reasons

Problem Formulation

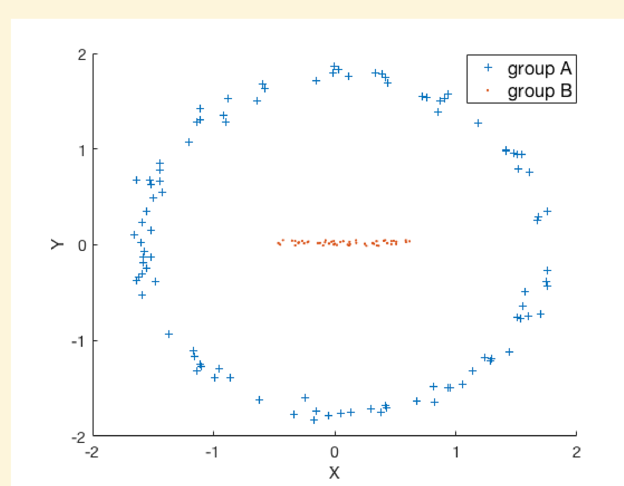
- $D = (A, B)$ is the data of two groups with rows as entries
- P is the orthonormal matrix for projection, needed to be found.

One may try to minimize the worst group's error:

$|D - DP|_F^2$ (standard PCA) → $\max\{|A - AP|_F^2, |B - BP|_F^2\}$ (after)



No single projection works for both groups



The above objective inappropriately ignores the more structured group

The above formulation ignores the inherent structures of each group. Let

$$\alpha = |A - AP_A|_F^2, \qquad \beta = |B - BP_B|_F^2$$

for best projections P_A, P_B separately of each group. α, β capture minimum error each group must have as a baseline. We optimize the **additional** error

$$\max\{|A - AP|_F^2 - \alpha, |B - BP|_F^2 - \beta\} \qquad \textbf{(Fair PCA)}$$

Algorithm

- Standard PCA can be solved efficiently by Singular Value Decomposition (SVD)
- Simple SVD can't solve fair PCA. But convex relaxation does extend.
- Convex relaxation finds $X \in \mathbb{R}^{n\times n}$ which behaves as PP^T for orthonormal $R \in \mathbb{R}^{n\times d}$. $\bar{\alpha}, \bar{\beta}$ are constants $|A|_F^2 - \alpha, |B|_F^2 - \beta$.

Convex Relaxation for Fair PCA

Input: data $A \in \mathbb{R}^{m_1\times n}, B \in \mathbb{R}^{m_2\times n}$ in n dimensions; target dimension $d \le n$

Algorithm: solve the following semidefinite program (SDP)

$$\min_{X\in\mathbb{R}^{n\times n}, z\in\mathbb{R}} z \text{ subject to}$$
$$z \ge \frac{1}{m_1}\left(\bar{\alpha} - \langle A^T A, X\rangle\right)$$
$$z \ge \frac{1}{m_2}\left(\bar{\beta} - \langle B^T B, X\rangle\right)$$
$$\mathrm{Tr}(X) \le d, 0 \preccurlyeq X \preccurlyeq I$$

This SDP is efferently solvable by multiplicative weight update (MW) method

$\hat{X} \in \mathbb{R}^{n\times n}$
$0 \preccurlyeq \hat{X} \preccurlyeq I, \mathrm{tr}(\hat{X}) \le d$

Problem: $\hat{X}$ has correct trace but higher rank than d

Fair-PCA Solution Rank Reduction

Input: data $A \in \mathbb{R}^{m_1\times n}, B \in \mathbb{R}^{m_2\times n}$ in n dimensions; target dimension $d \le n$; $\hat{X} \in \mathbb{R}^{n\times n}, \mathrm{tr}(\hat{X}) \le d$

Algorithm: apply SVD to $\hat{X}$

$$\hat{X} = \sum_{j=1}^{n} \hat{\lambda}_j u_j u_j^T$$

Solve the following linear program (LP)

$$\min_{\lambda\in\mathbb{R}^n, z\in\mathbb{R}} z \text{ subject to}$$
$$z \ge \frac{1}{m_1}\left(\bar{\alpha} - \textstyle\sum_{j=1}^n \lambda_j \langle A^T A, u_j u_j^T\rangle\right)$$
$$z \ge \frac{1}{m_2}\left(\bar{\beta} - \textstyle\sum_{j=1}^n \lambda_j \langle B^T B, u_j u_j^T\rangle\right)$$
$$\textstyle\sum_{j=1}^n \lambda_j \le d, 0 \le \lambda_j \le 1$$

Find one extreme solution λ^*. Set $X^* = \sum_{j=1}^n \lambda_j^* u_j u_j^T$

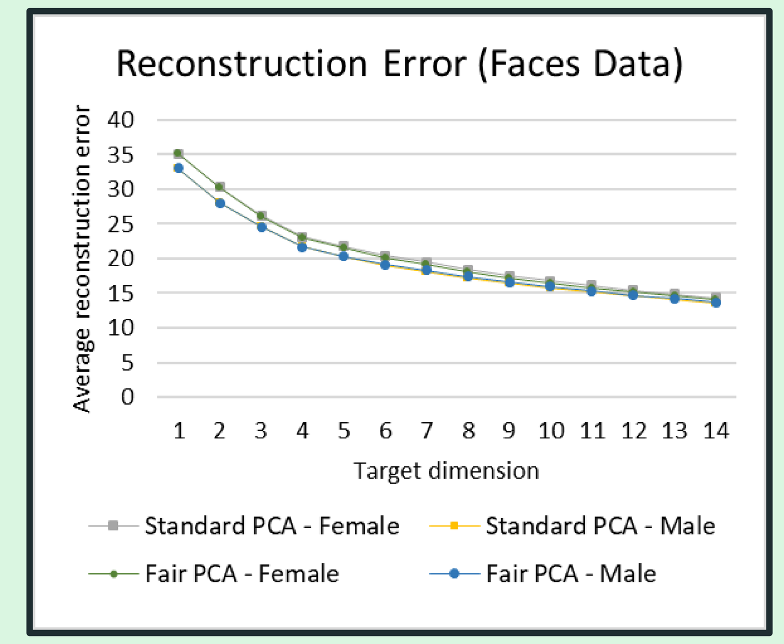
Provable Results

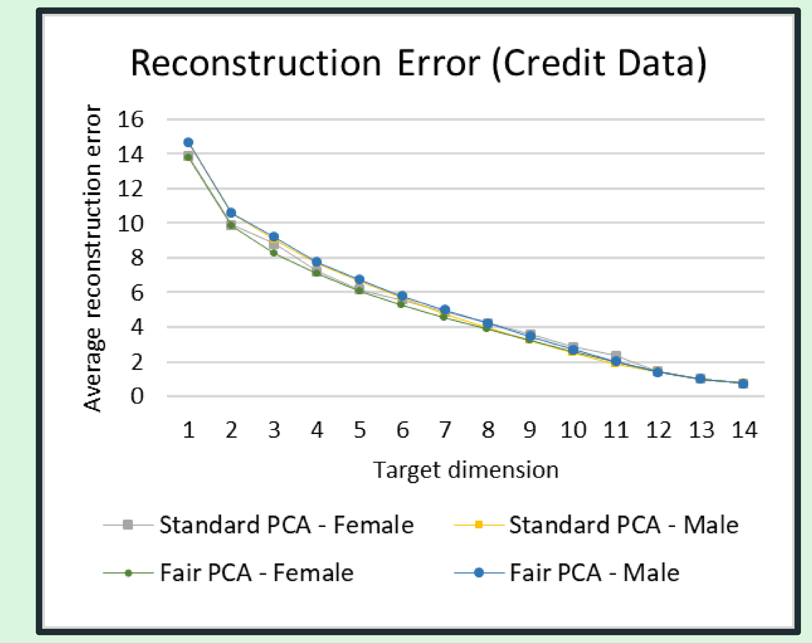
- X^* has rank at most $d + 1$.
- The final projection achieves optimal fair loss to both groups by adding one more dimension for representation

Technique

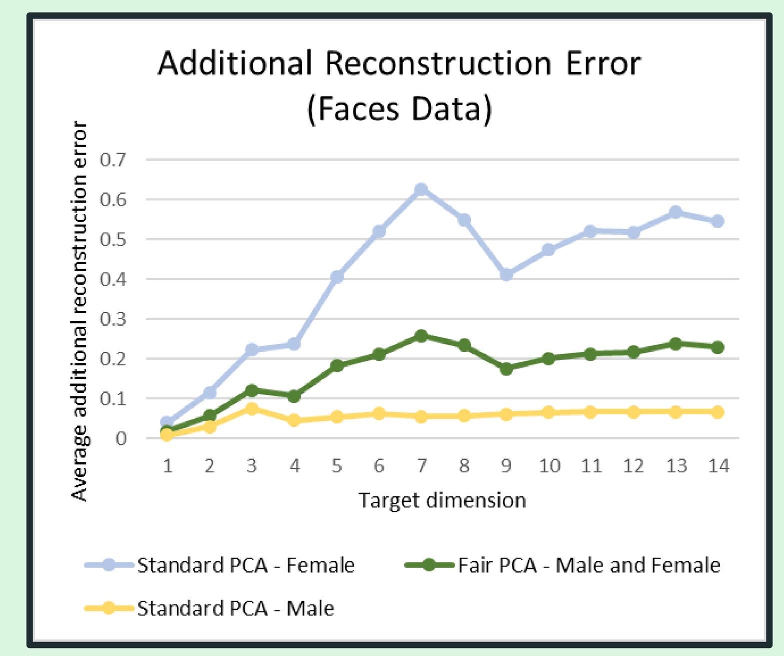
- Theory of extreme point of polyhedron

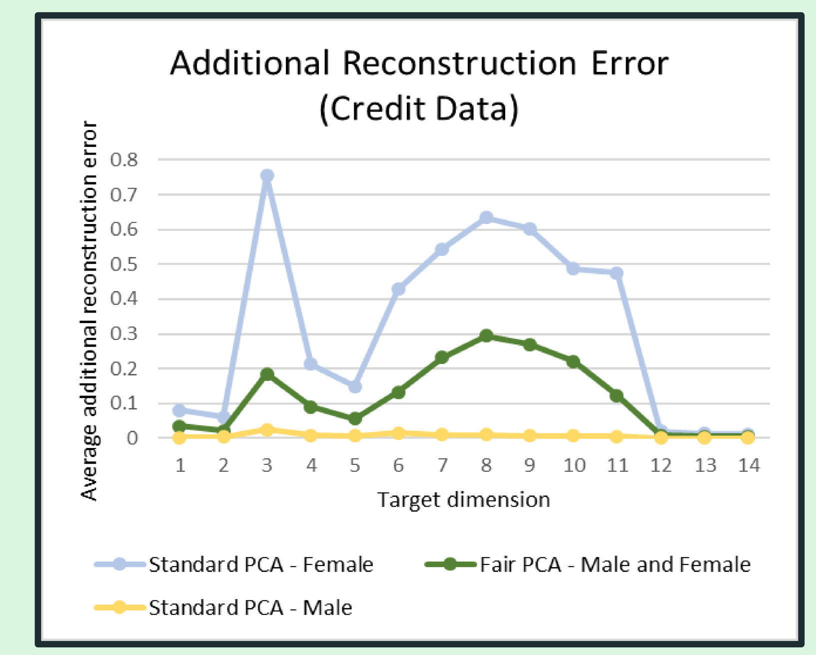
Experiments





The total reconstruction errors hide the significant **bias** of additional reconstruction errors





By using Fair PCA, our algorithm corrects the hidden bias

Dataset Specifications

Datasets	Size	Dimension	Runtime
Faces Data (images)	13k	42 x 42	
Credit Data	30k	21	

The runtime is from running our algorithm on standard single PC machine

Runtime

- The runtime of our algorithm is ≈10 times of applying SVD to the same problem. Much faster than predicted by theoretical analysis
- Very scalable to big data

Code: https://github.com/samirasamadi/Fair-PCA
Webpage: https://sites.google.com/site/ssamadi/fair-pca-homepage