Submodularity Helps in Nash and Nonsymmetric Bargaining

*Deeparnab Chakrabarty, Gagan Goel, Vijay V. Vazirani, Lei Wang*and Changyuan Yu*†

Abstract

Motivated by recent insights on the two solution concepts of Nash and nonsymmetric bargaining games, obtained from an algorithmic study of these notions [Vaz09], we take a fresh look at understanding their quality and robustness by subjecting them to several "stress tests".

Our tests are quite basic, e.g., we ask whether the solutions delivered are efficient, fair, respond in a desirable manner when agents change their disagreement points or play with a subset of the agents, and whether they lend themselves to efficient computatability.

Our main conclusion is that imposing *submodularity*, a natural economies of scale condition, on Nash and nonsymmetric bargaining games endows them with several desirable properties.

1 Introduction

Bargaining was first modeled as a game in John Nash's seminal 1950 paper [Nas50], using the framework of game theory given a few years earlier by von Neumann and Morgenstern [vNM44]. Bargaining is perhaps the oldest situation of conflict of interest, and since game theory develops solution concepts for negotiating in such situations, it is not surprising that this paper led to a theory (of bargaining) that lies today at the heart of game theory (e.g., see [Kal85, TL89, OR90, aum94]).

Nash's solution to the bargaining game is the unique point in the feasible set that satisfies 4 reasonable axioms. A well known generalization of Nash bargaining was obtained by Kalai [Kal77] by removing Nash's axiom of symmetry. Unlike the Nash bargaining solution, which is unique, a game has infinitely many nonsymmetric bargaining solutions. One can single out one of these by specifying the *clout* of each player; for the purposes of computation, we need to assume that the clouts of players are all rational, or equivalently integral, numbers. Kalai's main theorem implies that the nonsymmetric solution to an n-person bargaining game, with integral clouts, corresponds to a Nash bargaining solution to a game with a larger number of players which is obtained by replicating each of the n players as many times as her clout.

Over the years, the theory of bargaining developed along two major aspects. The more prominent one was the axiomatic approach whose goal was to characterize the solution concept that results from assuming a given set of axioms. The second was to determine the quality of the basic solution concepts of this theory by subjecting them to what may be viewed as "stress tests".

*College of Computing, Georgia Institute of Technology, Atlanta, GA 30332–0280

†Institute for Theoretical Computer Science, Tsinghua University, Beijing, 100084, P. R.China. This work was supported in part by the National Natural Science Foundation of China Grant 60553001, and the National Basic Research Program of China Grant 2007CB807900,2007CB807901.

These results determine whether or not a particular solution concept satisfies a particular test on *all instances* of the bargaining game.

We list the properties of Nash bargaining games that have been studied so far (e.g., see [Thob]). Thomson [Tho87] shows that all such games satisfy *monotonicity*, i.e., if the disagreement utility of a player is increased, then her utility cannot decrease. He also studied *strong monotonicity*, i.e., if the disagreement utility of one player is increased, then any other player's utility should not increase. He showed that Nash bargaining games do not always satisfy strong monotonicity. Thomson [Tho83] also defines the notion of *population monotonicity*, i.e., when bargaining with a proper subset of the players, an agent's utility should be non-decreasing, and shows that the Nash Bargaining solution doesn't satisfy this property.

Although useful, such an "all-or-nothing" results can give only limited information on the quality of a solution concept. In particular, the instances of interest may be highly structured and therefore may pass the test even though the entire solution concept does not. Motivated by recent insights on the two solution concepts of Nash and nonsymmetric bargaining games, obtained from an algorithmic study of these notions [Vaz09], we take a fresh look at understanding their quality and robustness. We deviate from the earlier line of work in two respects. First, we use the classification of these games given in [Vaz09], to determine if specific classes of these games satisfy a given property. Second, we define and study new tests that follow naturally from an algorithmic game theory perspective.

Our main conclusion is that imposing *submodularity*, a natural economies of scale condition, on Nash and nonsymmetric bargaining games endows them with several desirable properties. Our new tests are of a basic nature. For instance, we ask whether the solutions yielded by these two concepts are "efficient" and "fair", after giving reasonable, quantitative definitions of efficiency and fairness. We also define and study the notion of *surplus monotonicity*, i.e., if the disagreement utility of a player is increased, then her surplus utility, over and above the disagreement utility, increases.

As stated above, an important catalyst for our results is the insights gained from a deliberately *combinatorial* algorithmic approach to the Nash and nonsymmetric bargaining games insisted on in [Vaz09]. Besides the algorithms themselves, these include the combinatorial structural insights gained in obtaining the algorithms, and the classification of these games obtained by imposing restrictions on the the set of feasible utilities (or equivalently, imposing conditions on the constraints in the convex programs that capture the solutions to these games).

Bargaining is a basic mechanism for trading that has been used by mankind since times immemoriable. Indeed, one can even see portals on the Internet emerging that use this mechanism, e.g., iOffer.com is a website similar to eBay except that it uses bargaining instead of auctions to sell goods, and priceline.com now allows customers to name their price for airline tickets and hotel reservations. The Internet is particularly well suited for enabling individual negotiations to take place and one can expect a more wide-spread use of bargaining on it. Clearly, these new applications, involving both game theoretic and algorithmic issues, are best studied in the "traditional" style of algorithmic game theory.

1.1 Detailed description of our results

In order to describe our results more formally, we need to recall some classes of Nash and nonsymmetric bargaining games defined recently in [Vaz09]. LNB (Linear Nash and Nonsymmetric Bargaining Games) is the class of games whose feasible set of utilities is defined by finitely many linear constraints. UNB (Uniform Utility Nash and Nonsymmetric Bargaining Games) is the subclass of LNB in which for each available resource, each agent who uses this resource uses it in the same way, i.e., the linear constraints are all packing constraints and the coefficients in each constraint are 0/1. Clearly, only $2^{|A|}$ such constraints are needed, where A is the set of agents – one for each subset of A. We can now view the right hand sides of these constraints as being given by a set valued function over the power set of A. If this function is submodular, the game is said to be in the subclass SNB (Submodular Utility Nash and Nonsymmetric Bargaining Games) of UNB.

Several of our results show that a game in UNB has a certain property iff it is in SNB. Equivalently, these results characterize SNB within UNB. Submodularity is a natural economies of scale condition and the class SNB contains important natural games. These results should not come as a surprise – submodularity has been exploited in similar ways in the past in game theory, e.g., see Moulin and Shenker [MS97], showing that a cross-monotonic cost sharing method gives rise to a budget balanced and group strategyproof cost sharing mechanism iff the cost function is submodular, and the results of [JV07], giving analogous characterizations of Eisenberg-Gale markets defined via submodular functions.

We study *efficiency* of Nash bargaining games. A Nash bargaining game is said to be *utilitarian efficient* if and only if the total value of the Nash bargaining solution is equal to that of the most efficient solution that can be obtained in a centralized manner.

We will say that a Nash bargaining game is *strongly monotone* if whenever one player increases his disagreement utility, no other player's utility can increase in the resulting Nash or nonsymmetric bargaining solution.

We compare the max-min and min-max fairness of the Nash or nonsymmetric bargaining solution in relation to all feasible solutions that are Pareto optimal. The solution will be said to be *fully fair* if it satisfies both max-min and min-max fairness. This is the most stringent notion of fairness studied in literature.

We define the *surplus utility* of an agent to be the utility she gets in a Nash or nonsymmetric bargaining solution, over and above her disagreement utility. We say that a Nash or nonsymmetric bargaining game is *surplus monotone* if increasing the disagreement utility of any agent does not lead to an increase in her surplus utility.

Consider the utilities derived by an agent in the bargaining solution when playing with various subsets of the entire set of agents. We say that a Nash bargaining game is *population monotone* if the utility of any agent in the bargaining solution can only increase when playing with a subset of the agents. Similarly, we say that a nonsymmetric bargaining game is *population monotone* if the utility of any agent in the bargaining solution can only increase clouts of other agents are decreased; this notion is also independently defined by Thomson [Thoa].

For each of the properties stated above, except surplus monotonicity, we show that a game in UNB has this property iff it is also in SNB, i.e., these properties characterize SNB within UNB.

2 Nash and Nonsymmetric Bargaining Games

For a set of agents A, a Nash bargaining game is defined by a pair $(\mathbf{c}, \mathcal{P})$, where $\mathcal{P} \subseteq R_+^{|A|}$ is a compact and convex set which defines the feasible set of utilities of all the agents, and $\mathbf{c} \in \mathcal{P}$ is known as the disagreement point which defines the amount of utility each agent will get if the bargaining process fails.

Nash [Nas50] defined the bargaining solution $\mathbf{u}^* \in \mathcal{P}$ of this game to be the one which satisfies four axioms:

1. **Pareto optimality**: No point in $\mathcal{P}$ can weakly dominate $\mathbf{u}^*$.

2. **Invariance under transformation of utilities**: The affine transformation of the utilities leads to the same affine transformation on $\mathbf{u}^*$.

3. **Symmetry**: If the agents are renumbered, then it suffices to renumber the coordinates of $\mathbf{u}^*$.

4. **Independence of irrelevant alternatives**: If $\mathbf{u}^*$ is the solution for $(\mathbf{c}, \mathcal{P})$ where $\mathbf{c}, \mathbf{u}^* \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{P}$, then $\mathbf{u}^*(\mathbf{c})$ is also the solution for $(\mathbf{c}, \mathcal{S})$.

Nash proved that there is a unique point in $\mathcal{P}$ which satisfies these axioms, and moreover this point ($\mathbf{u} \in \mathcal{P}$) is the one that maximizes $\prod_{i \in A}(u_i - c_i)$ or equivalently $\sum_{i \in A} \log(u_i - c_i)$.

In the nonsymmetric bargaining game, each agent i has a positive bargaining weight or clout α_i which represents his relative bargaining power. The bargaining solution in this case is the point in the feasible region $\mathcal{P}$ which maximizes the objective function $\sum_{i \in A} \alpha_i \log(u_i - c_i)$.

2.1 Uniform Utility Nash and Nonsymmetric Bargaining Games

The class Linear Nash and Nonsymmetric Bargaining Games (LNB), defined in [Vaz09], consists of games whose feasible set $\mathcal{P}$ is defined by a finite number of linear constraints. The main focus of our paper will be on a natural subclass of LNB called Uniform Utility Nash and Nonsymmetric bargaining games (or UNB) [Vaz09]. In these games, the linear constraints are all packing constraints and in each of them, the coefficients of the variables are either 0 or 1. Clearly there can be at most $2^{|A|}$ such constraints, thus a function of the form $v : 2^A \rightarrow R^+$ uniquely encodes a feasible set in UNB games. A UNB game is called an SNB if the function v is a submodular function. In Section 3, we give examples of UNB and SNB games.

Now given a disagreement point* $\mathbf{c}$, and a fixed set of agents $T \subseteq A$, the solution to a uniform utility Nash bargaining game is captured by the following convex program:

$$\begin{aligned} \max \quad & \sum_{i \in T} \log(u_i - c_i) \\ s.t. \quad & \forall S \subset T : \sum_{i \in S} u_i \leq v(S) \\ & \forall i \in T : u_i \geq 0 \end{aligned} \tag{1}$$

For uniform utility nonsymmetric bargaining games, the objective functions that needs to be maximized is $\sum_{i \in T} \alpha_i \log(u_i - c_i)$.

For a fixed function $v : 2^A \rightarrow R^+$, we will define a family of games $F(v)$ to be the set of all Nash bargaining games for various choices of disagreement points $\mathbf{c}$ and set $T \subseteq A$. An instance $(\mathbf{c}, T) \in F(v)$ will refer to a particular Nash bargaining game in $F(v)$ with a fixed set T and disagreement point $\mathbf{c}$. We will use $u_i^*(\mathbf{c}, T)$ to denote the utility of the player i in the Nash's solution for the bargaining game $(\mathbf{c}, T) \in F(v)$.

*Throughout the paper, we will use boldface to denote vectors, however, when we wish to denote the coordinates we will revert to normal font. For instance, $\mathbf{c}$ is a vector, c_i is the ith coordinate, which is a scalar

Similarly, for a fixed function $v : 2^A \rightarrow R^+$ and a positive bargaining weight vector $\alpha = (\alpha_i)_{i \in A}$, we can define $F(v, \alpha)$ to be the set of all Nonsymmetric bargaining games for various choices of disagreement points $\mathbf{c}$ and set $T \subseteq A$.

We will assume that the following two natural conditions are satisfied by the function v :

1. **Non degenerate:** $v(\emptyset) = 0$.
2. **Non redundancy of sets:** $\forall S \subseteq A$, there exists a feasible utility vector $\mathbf{u}$ such that set S is *tight* w.r.t. $\mathbf{u}$, i.e. $\sum_{i \in S} u_i = v(S)$. It is easy to see (using duality) that this property is equivalent to the property that v satisfies fractional covering property i.e. $\forall S$, $v(S) \leq \sum_{B \subseteq A} v(B) x_B$, where $x_B{}'s$ are such that $\forall i \in S$: $\sum_{B : i \in B} x_B \geq 1$.

We will call such functions to be *valid* functions. Note that the second condition is without loss of generality as one can always modify the function v to satisfy this property without losing any of the feasible points. The second condition also implies 1)*Monotonicity:* for any $Z_1 \subset Z_2 \subseteq A$, we have $v(Z_1) \leq v(Z_2)$, and 2)*Complement freeness:* $v(Z_1 \cup Z_2) \leq v(Z_1) + v(Z_2)$.

2.2 Properties of UNB games

In this paper, we are interested in the following game theoretic properties of UNB games.

1. **Utilitarian Efficient**: For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is *Utilitarian Efficient* if $\min\limits_{(\mathbf{c},T) \in F(v)} u^*(c, T)$ is equal to $v(T)$, where $u^*(c, T) = \sum_i u_i^*(c, T)$.
2. **Fairness**: For any instance $I = (\mathbf{c}, T) \in F(v)$, define $core(I)$ to be the set of all feasible Pareto optimal solutions. For any vector $\mathbf{u}$, let $\mathbf{u}_{dec}$ be the vector obtained by sorting the components of $\mathbf{u}$ in decreasing order. A vector $\mathbf{x}$ *min-max dominates* $\mathbf{y}$ if $\mathbf{x}_{dec}$ is lexicographically smaller than $\mathbf{y}_{dec}$. Also let $\mathbf{u}^*$ be bargaining solution of instance I. Instance I is said to be *min-max fair* if the vector $\mathbf{u}^* - \mathbf{c}$ *min-max dominates* $\mathbf{y} - \mathbf{c}$ for all $\mathbf{y} \in core(I)$. $F(v)$ is said to be *min-max fair* if all the instances in $F(v)$ are min-max fair. Similarly, we define the notion of *max-min fairness.*
3. **Strong monotonicity**: For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is *strongly monotone* if, for all games in $F(v)$, the following property holds: On increasing the disagreement utility c_i of an agent i, the bargaining solution doesn't increase the utility for any other agent j, where $j \neq i$.
4. **Surplus monotonicity**: For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is *surplus monotone* if the surplus (i.e. $u_i^* - c_i$) of any agent i in the bargaining solution does not increase if her disagreement utility c_i increases. Formally, for any instance $(\mathbf{c}, T)$ and $(\mathbf{c}', T)$, $T \subseteq A$, where $\mathbf{c}' = \mathbf{c}$ except that $c_i' > c_i$, then we have
$$u_i^*(\mathbf{c}', T) - c_i' \leq u_i^*(\mathbf{c}, T) - c_i.$$
5. **Population monotonicity**: For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is *population monotone* if, for any $T_1 \subset T_2 \subseteq A$ and any agent $i \in T_1$, agent i cannot obtain more utility in the bargaining solution of instance $(\mathbf{c}, T_2)$ than in the bargaining solution of $(\mathbf{c}, T_1)$, i.e. $u_i^*(\mathbf{c}, T_1) \geq u_i^*(\mathbf{c}, T_2)$.

The main result of this paper is that for each of the above properties, other than surplus monotonicity, the property holds for $F(v)$ if and only if v is submodular.

All of the above definitions can be naturally extended for $F(v, \alpha)$ - the family of nonsymmetric bargaining games. Most of our results also extend naturally to this family of games; we point out the key differences in section 9. For the case of *population monotonicity*, one can instead consider a stronger version, which says that bargaining utility of an agent shouldn't decrease as the clout of some other agent is decreased. We again show that $F(v, \alpha)$ satisfy population monotonicity w.r.t clouts if and only if v is submodular.

Thus each of the above properties, other than surplus monotonicity, characterize SNB within UNB.

3 Examples of UNB and SNB games

We give few natural examples of UNB and SNB games here. Examples 1 illustrates a game which is in SNB; Example 2 illustrates a game which is in UNB but not in SNB; Example 3 illustrates a game which is in LNB but not in UNB. In each of these examples we only give the feasible set of utilities. The disagreement point $\mathbf{c}$ could be any point inside the feasible set.

Example 1 *(Sharing arcs of a network with single-source)* Consider a directed network $N = (V, A)$ with capacities on arcs. Let there be a source s in the network. Suppose there is a set of n agents A. Each agent controls a sink t_i and is interested in receiving flow from the source s. The utility of an agent i is the amount of flow that goes from source s to sink t_i. For any set S of agents, let $v(S)$ be the size of min cut separating source s and set S. Note that the function v is a submodular function [Meg74]. Now the feasible polytope of utilities is given by

$$\mathcal{P} := \{\forall\ S \subseteq A\ \sum_{i \in S} u_i \leq v(S);\ \forall i\ u_i \geq 0\}$$

Example 2 *(Branchings in a network)* Given a directed graph $G = (V, E)$, E is the set of resources with capacities on them. Agents are $A \subset V$. For an agent $s \in A$, let her desired object be branchings rooted at s and spanning all V, i.e., directed trees rooted at s and containing a path from s to each vertex in V. Suppose agent s sends a flow f_{sb} to each vertex in V using the branching object b, then his total utility is equal to the total flow sent along all his branchings, which is equal to $\sum_b f_{sb}$. For $S \subseteq A$, let $v(S)$ be the capacity of the minimum cut separating a vertex in $V - S$ from S. A result of [JV07] on the characterization of the feasible set of utilities shows that the game lies in (UNB\SNB), when the number of agents $|A| \geq 3$. Moreover, the feasible utility set is given via

$$\mathcal{P} := \{\forall\ S \subseteq A\ \sum_{i \in S} u_i \leq v(S);\ \forall i\ u_i \geq 0\}$$

Example 3 *(Sharing arcs of a network)* Consider a directed graph $G = (V, A)$ where the arcs have capacities and two agents who control two source-sink pairs (s_1, t_1) and (s_2, t_2) respectively. Utility of each agent i is the flow f_i that can be routed from s_i to t_i concurrently. Then the feasible utility set is known to be in (LNB\UNB) [CDV06]. To illustrate this, we give a small example here.

There are two arc-disjoint directed paths from s_1 to t_1 each of length 3 – (s_1, a, b, t_1) and (s_1, x, y, t_1), where a, b, x, y are nodes in the network; and there is a single directed path from s_2 to t_2 – (s_2, a, b, x, y, t_2). The only arcs that have capacities are (a, b) and (x, y) and each has a capacity of 1. Observe that when agent 2 sends x units of flow from s_2 to t_2, he "eats up" x units on *each* path of agent 1. Moreover, if agent 2 were absent, agent 1 could've sent a flow of 2 units (on each of its path) while agent 2 could send a maximum of 1 unit of flow only in the absence of agent 1. That is, the two agents do not share the resources in an uniform manner. The feasible polytope of utilities in this example is given by

$$\mathcal{P} := \{(u_1, u_2) : 0 \leq u_1 \leq 2,\ 0 \leq u_2 \leq 2;\quad u_1 + 2u_2 \leq 2\}$$

4 Preliminaries

For any valid function v, we say that S is *tight* w.r.t. $\mathbf{u}$ if $\sum_{i \in S} u_i = v(S)$. Let $\mathbf{u}^*$ be the solution to the convex program given in (1). Then by KKT conditions, there must exist variables $\{p_S,\ \forall S \subseteq T\}$ such that:

1. $\forall S \subseteq T,\ p_S \geq 0$.
2. $\forall S \subseteq T,\ p_S > 0 \Rightarrow \mathbf{u}^*$ makes set S tight.
3. $\forall k \in T$, we have $u_k > c_k \Rightarrow \sum\limits_{S: k \in S} p_S = \frac{1}{u_k^* - c_k}$. (or $\sum\limits_{S: k \in S} p_S = \frac{\alpha_k}{u_k^* - c_k}$ for the case of nonsymmetric bargaining game)

We will call p_S to be the price of set S.

We now show that for any set $T \subseteq A$ of agents, given any Pareto optimal point $\mathbf{u} \in \mathcal{P}$, there exists a disagreement vector $\mathbf{c}$ such that $\mathbf{u}$ is the bargaining solution for $(\mathbf{c}, T)$.

Lemma 4.1. *Given any valid function v, a set $T \subseteq A$ of agents, and a utility vector $\mathbf{u}$ with $u_i > 0, \forall i \in T$, $\mathbf{u}$ is Pareto optimal if and only if there exists a vector $\mathbf{c}$, with $c_i > 0\ \forall i \in T$, such that $\mathbf{u}$ is the bargaining solution for the instance $(\mathbf{c}, T)$.*

Proof. If $\mathbf{u}$ is a bargaining solution it has to be Pareto optimal. We now prove the converse. Let $\delta := \min_{i \in T} u_i > 0$. If $\mathbf{u}$ is Pareto optimal, we cannot increase u_i without changing other coordinates of $\mathbf{u}$. Therefore, every agent i is in (at least) one tight set $Z_i \subseteq T$. Set the price of any tight set $Z_i \subseteq T$ to be $p_{Z_i} = P$, where $P > \frac{1}{\delta}$. Let c_i's be defined as follows

$$c_i = u_i - \frac{1}{\sum_{Z \subseteq T : i \in Z} p_Z}, \tag{2}$$

We claim that the $\mathbf{u}, \mathbf{c}$ and the prices satisfy the KKT conditions implying $\mathbf{u}$ is the Nash bargaining solution with disagreement vector $\mathbf{c}$. This is because only tight sets are prices and c_i's are so defined to satisfy the second KKT condition. It remains to check $c_i > 0$ for all i. This is because every agent has at least one tight set and therefore

$$c_i > u_i - \frac{1}{\frac{1}{\delta}} = u_i - \delta = 0.$$

□

Now we give some properties of the submodular and non-submodular functions which will be used in our proofs.

Property 4.1. *Given a valid submodular function $v : 2^A \to R_+$, and a utility vector $\mathbf{u}$, if $Z_1, Z_2 \subseteq A$ are tight sets w.r.t. $\mathbf{u}$, then $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are also tight sets w.r.t. $\mathbf{u}$.*

Proof. $u(Z_1 \cup Z_2) + u(Z_1 \cap Z_2) = u(Z_1) + u(Z_2) = v(Z_1) + v(Z_2) \geq v(Z_1 \cup Z_2) + v(Z_1 \cap Z_2) \geq u(Z_1 \cup Z_2) + u(Z_1 \cap Z_2)$, where the second equality follows from the tightness of Z_1 and Z_2. □

By using the uncrossing argument and the above property, we get the following corollary.

Corollary 4.1. *Given any valid submodular function v, and $(c, T) \in F(v)$ (or $(c, T) \in F(v, \alpha)$ for the nonsymmetric case), we can choose the prices for all subsets of T in the KKT conditions, such that the tight sets with positive prices form a nested set family, i.e. $T = T_1 \supset T_2 \supset \cdots T_k \supset T_{k+1} = \emptyset$.*

We next state a property of non-submodular valid functions which enhances the following theorem of [CD09].

Theorem 4.1. *(Theorem 3.1 of [CD09]) Given any valid non-submodular function v, there exists a set $S \subset A$, $i, j \in A \setminus S$, $l \in S$ and a feasible utility vector $\mathbf{u}$ such that:*

1. *S, $S \cup \{i\}$, $S \cup \{j\}$ are all tight w.r.t $\mathbf{u}$.*
2. *No set containing both i and j is tight.*
3. *All tight sets containing either i or j, must contain the agent l with $u_l > 0$.*

[CD09] prove the above theorem by choosing the set S, i, j to be the minimal set which violate submodularity, that is, $v(S \cup i \cup j) + v(S) > v(S \cup i) + v(S \cup j)$. Since S is minimal, the restriction of v to the set $S \cup i$ (or $S \cup j$) is indeed submodular. Since v is valid, there is at least one utility vector $\mathbf{u}$ which makes S tight. Of these $\mathbf{u}$ is chosen which minimizes the number of tight subsets of S. u_i and u_j is defined so that $S \cup i$ and $S \cup j$ are tight, establishing part 1. Non-submodularity implies that the set $S \cup i \cup j$ is non-tight. [CD09] show that this is indeed true for any set containing both i and j establishing part 2. Establishing part 3 requires more work; [CD09] show that if there exist tight sets containing i and j which are disjoint, then one can show that the set $S \cup i \cup j$ is also tight which is not possible. We refer the interested reader to the paper for full details. We now extend the above theorem in a straightforward way.

Property 4.2. *Given any valid non-submodular function v, there exists a set $S \subset A$, $i, j \in A \setminus S$, $l \in S$ and a feasible utility vector $\mathbf{u}'$ such that:*

(a) *v is submodular on $S \cup \{i\}$.*

(b) *$S \cup \{i\}$, $S \cup \{j\}$ are both tight.*

(c) *Let $T = S \cup \{i, j\}$, $\mathcal{F}_k$ denotes the set of all subsets which contains k and are tight w.r.t $\mathbf{u}'$. We have*

$$\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j, \mathcal{F}_i \cap \mathcal{F}_j = \emptyset \quad .$$

(d) $u_k > 0$, $\forall k \in T$.

Proof. Let $S, i, j, \mathbf{u}$ be as in Theorem 4.1. As sketched in the proof above, S, i, j is chosen so that S is the minimal set contradicting submodularity. That is, v restricted to $S \cup i$ is submodular.

Note that condition (a) and (b) are satisfied. In part (c), we have $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ from part 2 of the above theorem. Part 3 implies $\mathcal{F}_i \cup \mathcal{F}_j \subseteq \mathcal{F}_l$; we need equality. Furthermore, part (d) might not be true since u_k could be 0 for some k.

We now modify the utility vector $\mathbf{u}$ to another vector $\mathbf{u}'$ such that $\mathbf{u}'$ satisfies our requirements. First of all, we show that we may assume without loss of generality that $u_k > 0$ for all $k \in S$. If there is any $r \in S$ such that $u_r = 0$, we remove the agent r from set S: let $S' = S \setminus \{r\}$. Note the element l in condition 3 of the above theorem remains in S'. Note that $S', i, j, l, \mathbf{u}$ still satisfies the conditions in Theorem 4.1. $S', S' \cup i, S' \cup j$ are tight since the total utility of these sets remain the same, and since the larger sets were tight, the valuations remain the same due to feasibility of $\mathbf{u}$ and monotonicity of v. Part 2 and 3 are easy to check. Thus, $u_k > 0$ for all $k \in S$. Also note that the validity of part (a) is retained since S' is a subset of S.

Now define $\mathbf{u}'$ as follows.

$$u'_i = u_i + \epsilon,\ u'_j = u_j + \epsilon,\ \text{and } u'_l = u_l - \epsilon, \quad u'_k = u_k \text{ for all } k \in T \setminus \{i, j, l\}$$

where ϵ is defined as follows to keep $\mathbf{u}'$ feasible.

$$\epsilon < \min\{\epsilon_0, u_l/2\},\ \text{where } \epsilon_0 := \min_{\text{nontight } Z \subseteq T} \frac{(v(Z) - \sum_{k \in Z} u_k)}{2}.$$

Since T is not tight, ϵ_0 is well defined and strictly positive. First we show that $\mathbf{u}'$ is feasible. For any set $Z \subseteq T$ which is not tight w.r.t $\mathbf{u}$, we have

$$\sum_{k \in Z} u'_k < \sum_{k \in Z} u_k + 2\epsilon_0 \leq v(Z),$$

, thus a non-tight set remains non-tight.

For any tight set $Z \subseteq T$ w.r.t $\mathbf{u}$, since we know that Z doesn't contain both i and j, there are two cases. If Z contains either i or j, then it contains l as well. Thus the total utility of that set doesn't change. Furthermore, the set remains tight. If Z doesn't contain i or j, then its total utility decreases implying feasibility.

We end by checking all conditions (a),(b),(c) and (d) are satisfied. Part (a) is not affected by definition of $\mathbf{u}'$ and is satisfied. Since tight sets containing i or j remain tight, part (b) remains true. Since non-tight sets remain non-tight, we have $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$. Since tight sets containing i and j remain tight, we have $\mathcal{F}_i \cup \mathcal{F}_j \subseteq \mathcal{F}_l$. Finally, any tight set containing l and not i or j becomes non-tight. So, we have equality in the above subset relation. Part (d) is true since $\mathbf{u}'_i, \mathbf{u}'_j > 0$ and for any $k \in S$, we have $\mathbf{u}'_k > 0$ as well. □

5 Utilitarian Efficient

We prove the following theorem.

Theorem 5.1. *For any valid function v, $F(v)$ is* Utilitarian Efficient *if and only if v is submodular.*

Proof. $\Leftarrow$: Suppose v is submodular. We want to show that for any disagreement point $\mathbf{c}$, and set $S \subseteq A$, if we restrict to the subproblem among agents in S, the Nash bargaining solution $\mathbf{u}^*$ satisfies $\sum_{i \in S} u^*_i = v(S)$.

Since $\mathbf{u}^*$ is the solution of Nash bargaining game, it must be Pareto optimal. Therefore every agent i is in some tight set T_i. Therefore by Property 4.1, we have $S = \cup_{i \in S} T_i$ is also tight, which means $\sum_{i \in S} u_i^* = v(S)$.

$\Rightarrow$: Suppose v is not submodular. By Property 4.2, there is a set $T = S \cup \{i, j\}$ and a feasible utility vector $\mathbf{u} = (u_k)_{k \in T}$ such that: (1) $u_k > 0, \forall k \in T$, (2) $S \cup i$ and $S \cup j$ are tight w.r.t. $\mathbf{u}$, (3) T is not tight w.r.t. $\mathbf{u}$. This is obtained from $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

Now for any $k \in T$, k is in some tight set w.r.t $\mathbf{u}$, hence by the lemma 4.1 in Section 4, there exists $\mathbf{c}$ such that $\mathbf{u}$ is the Nash bargaining solution corresponding to $\mathbf{c}$.

By condition 3 above, we have $\sum_{k \in T} u_k < v(T)$, which implies that it is not *Utilitarian Efficient.*

□

6 Fairness

In this section, we prove the following theorem.

Theorem 6.1. *For any valid function v, $F(v)$ is min-max and max-min fair if and only if v is submodular.*

Proof. $\Leftarrow$: Suppose v is submodular. Let $\mathbf{u}^*$ be the Nash bargaining solution for $(\mathbf{c}, T)$ where $T \subseteq A$. By corollary 4.1, we can choose the prices such that the tight sets w.r.t $\mathbf{u}^*$ with positive price form a nested set family, $T = T_1 \supset T_2 \supset ...... \supset T_t \supset \emptyset$. Since for any agent i, $u_i^* - c_i = 1/(\sum_{j: i \in T_j} p_{T_j})$, we see that $(\mathbf{u}^* - \mathbf{c})_{dec}$ has elements of $T_1 \setminus T_2$ followed by those in $T_2 \setminus T_3$ and so on. Moreover, agents in $T_j \setminus T_{j+1}$ have the same $u_i^* - c_i$.

Pick any element $\mathbf{g} \neq \mathbf{u}^*$ in $core((\mathbf{c}, T))$. Suppose $\mathbf{g} - \mathbf{c}$ min-max dominates $\mathbf{u}^* - \mathbf{c}$. Since $\mathbf{g}$ is Pareto optimal, every agent is in some tight set w.r.t $\mathbf{g}$. Hence by property 4.1, the whole set T is tight, $\sum_{k \in T} g_k = v(T)$. Since $\mathbf{g}$ is feasible, we also have

$$\sum_{k \in T_2} g_k \leq v(T_2)$$

Since T and T_2 are tight sets w.r.t $\mathbf{u}^*$, taking differences we get

$$\sum_{k \in T \setminus T_2} g_k \geq \sum_{k \in T \setminus T_2} u_k^* \tag{3}$$

Since each agent i in $T \setminus T_2$ has the highest $u_i^* - c_i$ among all the agents, if $\mathbf{g} - \mathbf{c}$ min-max dominates $\mathbf{u}^* - \mathbf{c}$, then for any $k \in T \setminus T_2$, we have $g_k \leq u_k^*$. Then by (3), we have $g_k = u_k^*, \forall k \in T \setminus T_2$. Similarly, we can show for any $1 \leq i \leq t$ and any $k \in T_i \setminus T_{i+1}$, $g_k = u_k^*$. Hence $\mathbf{g} = \mathbf{u}^*$ which is a contradiction.

This proof also shows that $\mathbf{u}^* - \mathbf{c}$ is the unique min-max fair utility vector. By using an argument similar to [JV08], we can show that any unique min-max fair utility vector is also max-min fair.

$\Rightarrow$: Suppose v is not submodular, then by property 4.2, there is a set $T = S \cup \{i, j\}$ and a $\mathbf{g} = (g_k)_{k \in T}$ such that: (1) $g_k > 0, \forall k \in T$, (2) $S \cup \{i\}$ and $S \cup \{j\}$ are tight w.r.t $\mathbf{g}$, (3) $\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

For each $k \in T$, let $c_k = g_k - \epsilon$, where $0 < \epsilon < \min_{k \in T}\{g_k\}$. Clearly $\mathbf{g}$ is a feasible core element corresponding to $\mathbf{c}$, since each k is in a tight set (either $S \cup \{i\}$ or $S \cup \{j\}$).

Let $\mathbf{u}^*$ be the Nash bargaining solution corresponding to $(\mathbf{c}, T)$. Note that by definition, $\mathbf{g} - \mathbf{c}$ is a vector with all entries equal to ϵ. Thus, since $\mathbf{g}$ is Pareto optimal, it is the unique min-max and max-min feasible solution. We now show that $\mathbf{g}$ cannot be the solution to the Nash bargaining game implying the solution, $\mathbf{u}^*$ is not fair.

If $\mathbf{g}$ were the solution, by KKT conditions, we can price all the subsets of T such that:

$$\frac{1}{g_l - c_l} = \sum_{Z \in \mathcal{F}_l} p_Z = \sum_{Z \in \mathcal{F}_i} p_Z + \sum_{Z \in \mathcal{F}_j} p_Z = \frac{1}{g_i - c_i} + \frac{1}{g_j - c_j}$$

which contradicts the fact that $g_l - c_l = g_i - c_i = g_j - c_j = \epsilon$.

□

7 Strong Monotonicity and Surplus Monotonicity

In this section we show that any SNB game $F(v)$ is both strongly monotone and surplus monotone. Moreover, we also show that any UNB game which is strongly monotone must be an SNB game. Let us first give an algorithm for finding the bargaining solution for a SNB game, i.e., an optimal solution to the convex program (1) when the function v is a submodular function. We will use properties of the final solution in the proof of theorem 7.1. (Note that Vazirani [Vaz09] also gave a polynomial time algorithm for SNB. Algorithm presented here is slightly different, and is needed to observe the required properties)

An algorithm for SNB:

1. The algorithm maintains a set of tight sets $\mathcal{T}$ which is initially empty. Also let $t = 0$.
2. while $T \notin \mathcal{T}$, increase t and let $y_i = t$ for all the free agents until some new set X gets tight. If X intersects with any set in $\mathcal{T}$, then since v is submodular, their union must be tight (Property 4.1). Pick X to be the maximal (inclusion-wise) tight set and put it in $\mathcal{T}$.

We have the following lemma:

Lemma 7.1. *The utility allocation returned by the above algorithm is an optimal solution to the convex program.*

Proof. The proof follows by giving dual values to the tight sets in $\mathcal{T}$ which satisfy the KKT conditions. In particular, if one considers the above algorithm as a continuous time algorithm, then if a set X goes tight at time t (i.e. the value of y at which the set X went tight), it is priced $1/t$ and all the maximal sets of $\mathcal{T}$ contained in X have their prices decreased by a factor $1/t$. Note that prices remain positive, and it can be easily checked that the KKT conditions hold. □

Theorem 7.1. *For any submodular valid function v, $F(v)$ is surplus monotone and strongly monotone.*

Proof. Suppose the disagreement of agent i goes from c_i to $c_i + \delta$. Call the new disagreement vector $\mathbf{c}'$. Let $f'(S) := v(S) - c'(S)$ for all S. To prove the theorem, it suffices to show that the optimum, $\mathbf{y}'$ of the convex program $\max\{\sum_i \log y_i : \; y(S) \le f'(S); \mathbf{y} \ge \mathbf{0}\}$ is dominated by $\mathbf{y}$, the solution to the original convex program with $f()$. For this shows that $u'_i - c'_i \le u_i - c_i$ implying rate monotonicity and for $j \neq i$, we have $u'_j - c'_j \leq u_j - c_j$ and full competitiveness follows from the fact that $c'_j = c_j$.

To show that $\mathbf{y}'$ dominates $\mathbf{y}$, we will use the continuous time algorithm presented above. Firstly, note that $f'(S) = f(S)$ for all sets not containing i and $f'(S) = f(S) - \delta$ for all others. This implies, that there is at least one agent j with $y'_j < y_j$. Secondly, observe from the description of the algorithm that for any agent j with $y'_j < y_j$, there must be a corresponding tight set in $\mathcal{T}'$ which contains both j and i.

We now show that if an agent j became non-free at time t in the original run (which means $y_j = t$), then by time t it must be in a tight set in the new run. We do so by showing that at time t if $y'_j = t$, then some set containing j at that time is tight (or over-tight which would imply $y'_j < t$).

Let A be the set containing j which went tight in the original run of the algorithm. Consider the set A in the new run of the algorithm at time t. Let $Q := \{k \in A : y'_k < y_k\}$. Note that if $j \in Q$, we are done. Assume $j \notin Q$. By the second observation made above and using the submodularity of v (to show union of intersecting tight sets is tight), we know there must exist a set Z which contains i and $Z \cap A = Q$, which is tight. That is, $y'(Z) = f'(Z) = f(Z) - \delta$. We claim that $y'(Z \cup A) \geq f'(Z \cup A)$ and thus we are done.

This is because

$$\begin{aligned} y'(Z \cup A) = y'(A \setminus Q) + y'(Z) &\geq y(A \setminus Q) + f'(Z) = y(A) - y(Q) + f'(Z) \\ &\geq f(A) - f(Q) + f(Z) - \delta \geq f(A \cup Z) - \delta = f'(A \cup Z) \end{aligned}$$

The first inequality follows from definition of Q, the second from the tightness of A under y and feasibility of y and the last follows from submodularity of f. □

Theorem 7.2. *If a valid function v is not submodular, then the UNB game $F(v)$ is not strongly monotone.*

Proof. Since v is not submodular, by property 4.2 there must exist a set S and agents $i, j \in A \setminus S$, $l \in S$, and a feasible utility vector $\mathbf{u}$ such that: (1) $S \cup \{i\}$, $S \cup \{j\}$ are both tight w.r.t. $\mathbf{u}$, (2)$\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$, (3) $u_k > 0$, $\forall k \in T$, where $T = S \cup \{i, j\}$.

We will now construct an instance $(\mathbf{c}, T) \in F(v)$ which is not strongly monotone. Let $\delta = \min_{k \in T} u_k > 0$. For tight sets $S \cup \{i\}$, $S \cup \{j\}$, we set their prices to be $p_{S,i}$, $p_{S,j}$ respectively, where $p_{S,i} = p_{S,j} = P = \frac{2}{\delta}$. For any other set $Z \subseteq T$, we set its price p_Z to be zero.

Let

$$\forall k \in T \quad c_k = u_k - \frac{1}{\sum_{Z \subseteq T, k \in Z} p_Z}$$

Since $S \cup \{i\}$ and $S \cup \{j\}$ are both tight, so for any $k \in T$, there exist at least one $Z \subseteq S$ such that $p_Z = P$, and we have

$$c_k \geq u_k - \frac{\delta}{2} > 0.$$

By the definition of $\mathbf{c}$, all the KKT conditions hold, thus $\mathbf{u}$ is the bargaining solution w.r.t. $(\mathbf{c}, T)$.

Suppose there exists a $\mathbf{c}'$ and a corresponding bargaining solution $\mathbf{u}'$, such that: (1) $\forall k \in T, k \neq j, c'_k \geq c_k$, and (2) $c'_j = c_j$ and $u'_j > u_j$.

Using this, we can show that there exists a game in F(v) which is not strongly monotone. This is because $\mathbf{c}'$ can be obtained from $\mathbf{c}$ by increasing only the coordinates other than j. If $F(v)$ is strongly monotone, then each time a coordinate of $\mathbf{c}$ is increased utility allocated to j shouldn't increase. But if $u'_j > u_j$ is true then we get a contradiction.

Let $\mathbf{u}'$ be same as $\mathbf{u}$ except that $u'_j = u_j + \epsilon$, $u'_i = u_i + \epsilon$, $u'_l = u_l - \epsilon$. Using arguments similar to the proof of property 4.2, one can show that there exists small enough ϵ (given below) so that $\mathbf{u}'$ is feasible.

$$\epsilon < \min\{\epsilon_0, u_l/2\}, \text{ where } \epsilon_0 := \min_{\text{non-tight } Z\subseteq T} \frac{(v(Z) - \sum_{k\in Z} u_k)}{2}.$$

Now we will construct $\mathbf{c}'$ satisfying the above mentioned conditions, and *price* the sets so that along with $\mathbf{u}'$ they satisfy the KKT conditions, which would imply that $\mathbf{u}'$ is the bargaining solution for the disagreement point $\mathbf{c}'$.

We assign positive price to sets $S\cup i$, $S\cup j$ only, say $p'_{S,i}$ and $p'_{S,j}$ respectively, which is consistent with second KKT condition as both the sets are tight under $\mathbf{u}'$. Third KKT condition says that: 1)$c'_i = u'_i - \frac{1}{p'_{S,i}}$, 2)$c'_j = u'_j - \frac{1}{p'_{S,j}}$, and 3)$c'_k = u'_k - \frac{1}{p'_{S,i}+p'_{S,j}}$, $\forall k \in T, k \neq i, j$. Thus to get $\mathbf{c}'$, one can equivalently find prices $p'_{S,i}$ and $p'_{S,j}$.

Now, since we want $c'_j = c_j$, we get that

$$u'_j - \frac{1}{p'_{S,j}} = u_j + \epsilon - \frac{1}{p'_{S,j}} \quad \Rightarrow \quad \epsilon - \frac{1}{p'_{S,j}} = -\frac{1}{p_{S,j}}$$

Similarly, for $k \neq j$, we want $c'_k \geq c_k$. Expanding c'_k and c_k for different k, we get that following necessary condition should hold

$$-\epsilon - \frac{1}{p'_{S,i} + p'_{S,j}} \geq -\frac{1}{p_{S,i} + p_{S,j}}$$

It is not difficult so see that one can find $p'_{S,i}$ and $p'_{S,j}$ which satisfies above two conditions as long as the ϵ is choosen such that $\epsilon < \frac{1}{p_{S,i}+p_{S,j}} = \frac{\delta}{4}$.

To sum up, by setting $\epsilon = \min\{\epsilon_0/2, \delta/8\}$, we can find $p'_{S,i}, p'_{S,j}$ such that:

$$p'_{S,j} = \frac{1}{\epsilon + \frac{1}{p_{S,j}}}, \; p'_{S,i} \geq \frac{1}{\frac{1}{p_{S,i}+p_{S,j}} - \epsilon} - p'_{S,j}$$

Note that this value of ϵ is consistent with the previous mentioned upper bound on it. Therefore, we can construct $\mathbf{c}'$ such that $\mathbf{u}'$ is the bargaining solution w.r.t. $\mathbf{c}'$ and $c'_k \geq c_k$, $\forall k \in T$, $c'_j = c_j$. Thus $(\mathbf{c}, T) \in F(v)$ is not strongly monotone. □

However, SNB games are not the only UNB games which are surplus monotone. The following is an example of a UNB game which is not an SNB game but is still surplus monotone. It is an interesting open question to characterize surplus monotone games based on their valuation functions.

Example 4. Consider the following game with three agents, $A = \{1, 2, 3\}$, and $v : 2^A \rightarrow R^+$ is defined as: $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 3$, $v(\{1, 2\}) = v(\{2, 3\}) = v(\{3, 1\}) = 4$, $v(\{1, 2, 3\}) = 6$. This game is not an SNB game. However rate monotonicity holds. To prove this rigorously, one needs to do a case analysis. We sketch how this is done. It is not hard to see any bargaining solution either has $u_1 = u_2 = u_3 = 2$, or there is an agent, say agent 1, with utility between 2 and 3. In the former case, the sets $\{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}$ are tight and can have positive price. In the latter case, however, all tight sets contain the agent 1. Now suppose the disagreement of some agent j is increased and for contradiction's sake, assume that the difference $(u_j - c_j)$ also increases. Note, by the KKT conditions this is equivalent to saying that the price "faced" by agent j $(\sum_{S:j\in S} p_S)$ decreases. This cannot happen, and one can argue by going over all cases. We do one such case. Suppose agent j had utility between 2 and 3 (and thus all tight sets contain j). When c_j is increased and rate monotonicity is violated, u_j also increases. Thus, u_k decreases for all $k \neq j$ (since the only tight set containing $k \neq j$ contains j). Since c_k is the same, the price faced by k increases. Thus, with the new disagreements, the total price faced by $k \neq j$ increases, but total price faced by j decreases; this is not possible since all tight sets (priced sets) contain j. One can do the other cases similarly.

8 Population Monotonicity

In this section, we investigate the population monotonicity in a UNB game. Generally speaking, population monotonicity means that when bargaining with a superset of agents, one cannot obtain more utility. This seems to be a reasonable property of bargaining problem, however, it turns out that the population monotonicity only holds in SNB games.

Theorem 8.1. *For any valid function v, $F(v)$ is population monotone if and only if v is submodular.*

Proof. $\Leftarrow$: The proof of this is similar to the proof that SNB games are strongly monotone. Once again, we use the continuous time algorithm of Section 7 to prove that for submodular v, $F(v)$ is population monotone and urge the readers to recall the same.

Consider the run of the algorithm when the set of agents is T_1 and T_2. Let the utility vectors obtained be $\mathbf{u}^1$ and $\mathbf{u}^2$ respectively and let $\mathbf{y}^1 := \mathbf{u}^1 - \mathbf{c}(T_1)$ and $\mathbf{y}^2 := \mathbf{u}^2 - \mathbf{c}(T_2)$. It is enough to show that for all agents in T_1, $y_i^1 \geq y_i^2$. Pick an agent $i \in T_1$ and suppose in the run of the algorithm with T_1, at time t, i becomes non-free – that is, $y_i^1 = t$. We now show that in the run of the algorithm with T_2, there exists a subset of T_2 containing i which is tight by time t (or over tight, which would imply $y_i^2 < t = y_i^1$).

Let A be the first set containing i which goes tight in the run of the algorithm with T_1. Let $Q \subseteq A := \{j \in A : y_j^2 < y_j^1\}$. We may assume that $i \notin Q$ for otherwise we are done. Moreover, for any $j \in Q$, $y_j^2 < y_j^1$ implies that there must be a set $Z_j \subseteq T_2$ which goes tight in the run of the algorithm with T_2 and Z_j is not a subset of T_1. Let Z be the union of all these sets $\{Z_j : j \in Q\}$. Note that $Q = Z \cap A$ and that Z is tight.

Consider the set $Z \cup A$ – we have

$$\begin{aligned} y^2(Z \cup A) = y^2(Z) + y^2(A \setminus Q) &\geq f(Z) + y^1(A \setminus Q) \\ &= f(Z) + y^1(A) - y^1(Q) \geq f(Z) + f(A) - f(Q) \geq f(Z \cup A) \end{aligned}$$

where the last but one inequality follows from the fact that A was tight w.r.t. $\mathbf{y}^1$ and $\mathbf{y}^1$ was feasible, and the last follows from submodularity of f.

$\Rightarrow$: If v is not submodular, by property 4.2, there exists a set $S \subseteq A$, $l \in S$, $i, j \in A \setminus S$ and a feasible utility vector $\mathbf{u}$ such that the following hold:

1. $S \cup \{i\}$ and $S \cup \{j\}$ are both tight w.r.t $\mathbf{u}$.

2. $\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$ and $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

3. v is submodular on $S \cup \{i\}$.

4. for any $k \in T$, $u_k > 0$, where $T = S \cup \{i, j\}$.

For this utility $\mathbf{u}$, we can find $\mathbf{c}$ such that $\mathbf{u}$ is the bargaining solution of instance $(\mathbf{c}, T)$, i.e. $\mathbf{u} = \mathbf{u}^*(\mathbf{c}, T)$. We will show that after we remove agent j, there must be some agent in $S \cup \{i\}$ such that her utility in the bargaining solution of instance $(\mathbf{c}, S \cup \{i\})$ will be smaller.

Consider the instance $(\mathbf{c}, S \cup \{i\}) \in F(v)$, let the bargaining solution be $\mathbf{u}' = \mathbf{u}^*(\mathbf{c}, S \cup \{i\})$. Since v is submodular on $S \cup \{i\}$, we know that $S \cup \{i\}$ is tight w.r.t $\mathbf{u}'$. Recall that $S \cup \{i\}$ is also tight w.r.t $\mathbf{u}$, we have:

$$\sum_{k \in S \cup \{i\}} u_k = \sum_{k \in S \cup \{i\}} u'_k$$

If no agent's utility differs in two bargaining solutions, the set of tight sets does not change.Consider $r_l := \frac{1}{u_l - c_l}$, in the solution of instance $(\mathbf{c}, T)$:

$$r_l = \sum_{Z \in \mathcal{F}_l} p_Z = \sum_{Z \in \mathcal{F}_i} p_Z + \sum_{Z \in \mathcal{F}_j} p_Z = r_i + r_j$$

After we remove the agent j, since $\mathcal{F}_j$ is removed from $\mathcal{F}_l$, i.e. $\mathcal{F}'_l = \mathcal{F}'_i$, thus we have $r_l = r_i$, which is a contradiction. So at least one agent's utility differs. Since

$$\sum_{k \in S \cup \{i\}} u_k = \sum_{k \in S \cup \{i\}} u'_k$$

there must be an agent $k \in S \cup \{i\}$ with $u'_k < u_k$.

Therefore, if any UNB game $F(v)$ is population monotone, v must be submodular.

□

9 Extension to Nonsymmetric Nash Bargaining Games

Recall that $F(v, \alpha)$ is a family of nonsymmetric bargaining games, where function $v : 2^A \to R^+$ encodes the feasible set and $\alpha = (\alpha_i)_{i \in A}$ specifies the bargaining weights or clouts of all the agents.

Theorem 9.1. *For any valid function v and positive vector α, we have the following:*

1. *$F(v, \alpha)$ is utilitarian efficient iff v is submodular.*

2. *$F(v, \alpha)$ is strongly monotone iff v is submodular.*

3. *$F(v, \alpha)$ is max-min fair and min-max fair iff v is submodular.*

4. *$F(v, \alpha)$ is surplus monotone.*

5. *$F(v, \alpha)$ is population monotone w.r.t clouts iff v is submodular*

Proof. All the proofs can be easily extended, and we only give key modifications here.

(1) In equation (2), which is based on the KKT conditions, we do the following modification:

$$c_i = u_i - \frac{\alpha_i}{\sum_{Z \subseteq T : i \in Z} P_Z},$$

so that the lemma 4.1 still holds for the nonsymmetric case. Rest of the proof follows in similar fashion.

(3) For the case of fairness, we need to redefine the notion of fairness taking into account the relative bargaining powers of different agents. A bargaining game is said to be *min-max* (or *max-min*) fair if the vector $< \frac{u_i^* - c_i}{\alpha_i} >$ dominates $< \frac{x_i - c_i}{\alpha_i} >$ for all vectors $\mathbf{x}$ in the core. Proof follows on the same lines as the one given in section 6

(5) The proof for one direction, namely $F(v, \alpha)$ is *population monotone w.r.t. clouts* when v is submodular, follows from the algorithm and the arguments used in section 7. For the other direction, we use the same examples as given in [CD09] and keep the disagreement vector to be the $\mathbf{0}$ vector.

□

10 Discussion

Many of our criteria, e.g., for efficiency or fairness, are the most stringent possible, i.e., characterizing games having *full* efficiency or max-min and min-max fairness. There is probably much to be gained by considering relaxed notions of these properties.

A more specific problem is that we have proved surplus monotonicity for games in SNB and have given examples to show that this does not characterize SNB within UNB. We leave the open problem of characterizing the family of games within UNB that are surplus monotonic.

We note that the study of these properties is not applicable to Kalai-Samaridinko (KS) bargaining solution concept [KS75] , as the KS solution is not well defined for UNB games when the number of players is more than two [Rot79] [†]

11 Acknowledgement

We wish to thank Ehud Kalai and William Thomson for valuable discussions and pointers into the literature.

References

[aum94] Cooperative models of bargaining. In R. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications*, pages 1237–1284, 1994.

[CD09] D. Chakrabarty and N. Devanur. On competitiveness in uniform utility allocation markets. In *Operations Research Letters*, 2009.

[CDV06] D. Chakrabarty, N. Devanur, and V. V. Vazirani. New results on rationality and strongly polynomial solvability in eisenberg-gale markets. In *WINE*, 2006.

[JV07] K. Jain and V. V. Vazirani. Eisenberg-Gale Markets: Algorithms and Game-Theoretic Properties. In *Proceedings of ACM Symposium on Theory of Computing*, 2007. To appear in *Games and Economic Behavior*.

[†] Although Roth [Rot79] didn't explicitly mention that his impossibility example is in UNB, it is clear from the proof, and is also not difficult to show otherwise.

[JV08] K. Jain and V. V. Vazirani. Equitable cost allocations via primal-dual-type algorithms. *SIAM Journal on Computing*, 38(1):241–256, 2008.

[Kal77] E. Kalai. Nonsymmetric Nash solutions and replications of 2-person bargaining. *International Journal of Game Theory*, 6:129–133, 1977.

[Kal85] E. Kalai. Solutions to the bargaining problem. In L. Hurwicz, D. Schmeidler, and H. Sonnenschein, editors, *Social Goals and Social Organization*, pages 75–105. Cambridge University Press, 1985.

[KS75] E. Kalai and M¿ Smorodinsky. Other solutions to nash's bargaining problem. *Econometrica*, 43(3):513–518, 1975.

[Meg74] N. Megiddo. Optimal flows in networks with multiple sources and sinks. *Mathematical Programming*, 7:97–107, 1974.

[MS97] H. Moulin and S. Shenker. Strategyproof sharing of submodular costs: Budget balance versus efficiency. *Economic Theory*, 1997.

[Nas50] J. F. Nash. The bargaining problem. *Econometrica*, 18:155–162, 1950.

[OR90] M. Osborne and A. Rubinstein. *Bargaining and Markets*. Academic Press, Inc., 1990.

[Rot79] A. E. Roth. An impossibility result concerningn-person bargaining games. *International Journal of Game Theory*, 8(3):129–132, 1979.

[Thoa] William Thomson. Anonymity and population-monotonicity. *Mimeo, 1994*.

[Thob] William Thomson. Bargaining and the theory of cooperative games: John Nash and Beyond. *Manuscript, 2008. To appear*.

[Tho83] William Thomson. The fair division of a fixed supply among a growing population. *Mathematics of Operations Research*, 8(3):319–26, August 1983.

[Tho87] William Thomson. Monotonicity of bargaining solutions with respect to the disagreement point. *Journal of Economic Theory*, 42(1):50–58, June 1987.

[TL89] W. Thomson and T. Lensberg. *Axiomatic Theory of Bargaining With a Variable Population*. Cambridge University Press, 1989.

[Vaz09] V. V. Vazirani. Nash bargaining via flexible budget markets. *Submitted*, 2009.

[vNM44] J L von Neumann and O Morgenstern. *Theory of Games and Economic Behavior*. 1944.