ALLOCATION OF DIVISIBLE GOODS UNDER LEXICOGRAPHIC PREFERENCES

LEONARD J. SCHULMAN AND VIJAY V. VAZIRANI

ABSTRACT. We present a simple and natural non-pricing mechanism for allocating divisible goods among strategic agents having lexicographic preferences. Our mechanism has favorable properties of strategy-proofness (incentive compatibility). In addition (and even when extended to the case of Leontief bundles) it enjoys Pareto efficiency, envy-freeness, and time efficiency.
1. Introduction

The study of principled ways of allocating divisible goods among agents has long been a central topic in mathematical economics. The method of choice that emerged from this study, the Arrow-Debreu market model [1], provides a powerful approach based on pricing and leads to the fundamental welfare theorems. However, these market-based methods have limitations when agents are assumed to be strategic, e.g., these methods are not incentive compatible. Issues of the latter kind have been studied within the area of mechanism design for the last four decades, and have played a large role in the last decade in algorithmic game theory [20].

In this paper our primary focus is a particular simple and natural non-pricing mechanism, the Synchronized Greedy (SG) mechanism, for allocating divisible goods. The SG mechanism generalizes a mechanism introduced by Crès and Moulin [7] in the context of a job scheduling problem, and studied further by Bogomolnaia and Moulin [6] for the allocation of indivisible goods. (These mechanisms for allocation of indivisible goods are randomized. Our focus on divisible goods is just as general, since an allocation of divisible goods can be used without further modification as a randomized allocation of indivisible goods in the same quantities.) For the setting defined below, we show that SG has favorable efficiency, incentive compatibility, and fairness properties. Our setting assumes that each agent has a lexicographic preference relation over goods. We note that this preference relation is rational in the sense that it is complete and transitive. It does not, on the other hand, satisfy the continuity condition that preferences between allocations are preserved under limits; a rational preference relation that also satisfies this continuity condition is known to be representable by a utility function, see [17].

What favorable properties can be achieved in the area of goods allocation using only non-pricing mechanisms is a difficult question. The present paper can be regarded as carving out a certain special case, namely the limit in which agents have very strong preferences among the goods, and providing strong positive guarantees in this case. The limit of strong preferences is naturally captured by lexicographic preferences. In this limit there is an additional motivation to use non-pricing mechanisms, because very strong preferences might cause a pricing mechanism to do little more than ensure that the wealthiest agents get what they want. By focusing on non-pricing mechanisms, we can study what game-theoretic properties an allocation mechanism can achieve, without depending on what resources the agents have (or care to invest) in the game.

There are many every-day examples where something like our model comes up—naturally, not in market economy transactions, but in other societal mechanisms for allocation. An important class is allocation of public resources, e.g., placement lotteries in public schools, and see Kojima [15] for further examples and references. (Note also that this kind of example employs a standard reduction of the indivisible goods case to the divisible goods case by randomization.) Another class of examples is coordination within teams. To illustrate, suppose Alice, Bob and Carol are camp counselors. They have to divide up many tasks in their 12 daily duty hours: leading canoeing activities, leading rock-climbing, meeting parents of prospective campers, kitchen duty, cleaning latrines, etc. Alice loves to rock-climb, but fears the water and prefers even latrine duty to canoeing. Bob loves canoeing, but will do anything to stay out of the kitchen where he once saw a rat. Carol enjoys cooking even for large groups, but cannot stifle her impatience with visiting parents. These modestly-paid counselors are not going to arrange their shared summer schedule with a monetary exchange. But they might use a protocol like the one we analyze in this article. Due to its strategy-proofness it can be implemented simply and transparently.

The broader challenge of the utility-functions version of the allocation problem remains largely open. The simplicity of the SG mechanism is perhaps encouraging toward the existence of allocation mechanisms maintaining favorable (maybe weaker) game-theoretic properties in this setting.
Parameters of the problem. In the allocation problem there are \( m \) distinct divisible goods which need to be allocated among \( n \) agents. Good \( j \) (\( 1 \leq j \leq m \)) is available in the amount \( q_j > 0 \), and agent \( i \) (\( 1 \leq i \leq n \)) is to receive a specified \( r_i > 0 \) combined quantity of all goods; the parameters satisfy \( \sum_j q_j \geq \sum_i r_i \), i.e., the total supply is at least as large as the total demand. If this inequality fails, our mechanism may still be run after rescaling expectations so that each agent \( i \) is to receive the quantity \( r'_i = r_i (\sum q_j) / (\sum r_k) \). So in the sequel we may assume \( \sum_j q_j \geq \sum_i r_i \).

Preferences: the non-Leontief case. The non-Leontief case of our problem is this. An allocation of goods is a list of numbers \( a_{ij} \geq 0 \), with \( \sum_j a_{ij} = r_i \) and \( \sum_i a_{ij} \leq q_j \), indicating that agent \( i \) receives quantity \( a_{ij} \) of good \( j \). The vector \( a_{i*} = (a_{i1}, \ldots, a_{im}) \) is referred to as agent \( i \)'s (share of the) allocation. Each agent \( i \) has a preference list, which is a permutation \( \pi_i \) of the goods; \( (a_{i\pi_i(1)}, \ldots, a_{i\pi_i(m)}) \) is agent \( i \)'s sorted allocation. Agent \( i \)'s preference among allocations is induced by lexicographic order. That is to say, agent \( i \) lexicographic-prefers \( a_{i*} \) to \( b_{i*} \), written \( a_{i*} >_i b_{i*} \), if the leftmost nonzero coordinate of \( (a_{i\pi_i(1)}, \ldots, a_{i\pi_i(m)}) - (b_{i\pi_i(1)}, \ldots, b_{i\pi_i(m)}) \) is positive. Furthermore, we will say that agent \( i \) majorization-prefers \( a_{i*} \) to \( b_{i*} \), written \( a_{i*} >_M b_{i*} \), if

\[
\text{for all } k = 1, \ldots, m : \sum_{\ell=1}^k a_{i\pi_i(\ell)} \geq b_{i\pi_i(\ell)},
\]

with at least one of the inequalities being strict. The symbols \( \geq_i \) and \( >_i \) will have the obvious interpretations.

Since an agent's preferences depend only on his own share of the allocation, we speak interchangeably of an agent's preference for an allocation or an allocation share. In particular, \( a_{i*} >_i b_{i*} \) may be written more simply as \( a >_i b \), and \( a_{i*} >_M b_{i*} \) may be written as \( a >_M b \).

Preferences: Leontief Bundles. Some of our results hold in the more general setting of lexicographic preferences among Leontief bundles, and some fail in that setting; details below. A Leontief bundle is specified by a nonnegative vector \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \) (where \( \mathbb{R}^m_+ = \mathbb{R}^m \setminus \{0\} \) are nonnegative reals). The set of goods \( j \) for which \( \lambda_j \) is positive is called the support of this bundle. If \( q \in \mathbb{R}_+^m \) then the bundle \( \lambda \) may be allocated from \( q \) in any quantity \( \alpha \in \mathbb{R}_+^m \) such that \( \alpha \lambda_j \leq q_j \) for all \( j \). In an instance of our problem, a list of \( M \) Leontief bundles \( \lambda_1, \ldots, \lambda_M \) is specified, including among them the \( m \) singleton bundles (hence always \( M \geq m \)). It is convenient, and in our context sacrifices no generality, to impose the convention that for every bundle \( \lambda = (\lambda_1, \ldots, \lambda_m) \), \( \sum_1^m \lambda_j = 1 \).

The case \( M = M \), in which all bundles are singletons, is of course a special case of the Leontief framework, but to distinguish it from the general situation we call it the "non-Leontief" case.

The framework we are concerned with is that each agent \( i \) has a preference list specified by a permutation \( \pi_i \) of the bundles. A Leontief allocation is an \( n \times M \) matrix \( \ell \) in which \( \ell_{ik} \) represents the quantity of bundle \( k \) allocated to agent \( i \). A Leontief allocation \( \ell \) imposes the goods allocation \( A(\ell) \), an \( n \times m \) matrix, by \( A(\ell)_{ij} = \sum_{k=1}^M l_{ik} \lambda^k_j \). We further require that a Leontief allocation satisfy the conditions \( \sum_j A(\ell)_{ij} = r_i \) (thanks to the convention above this is equivalent to \( \sum_k l_{ik} = r_i \)) and \( \sum_i A(\ell)_{ij} \leq q_j \). We speak of \( A(\ell)_{i*} \) and \( l_{i*} \) as agent \( i \)'s share of, respectively, the goods and the Leontief bundles. The vector \( (l_{i\pi_i(1)}, \ldots, l_{i\pi_i(M)}) \) is agent \( i \)'s sorted Leontief share. Agent \( i \)'s preference among allocations is induced by lexicographic order on his share of the allocation. That is to say, agent \( i \) lexicographic-prefers \( \ell \) to \( \ell' \), written \( \ell >_i \ell' \), if the leftmost nonzero coordinate of \( (l_{i\pi_i(1)}, \ldots, l_{i\pi_i(M)}) - (l'_{i\pi_i(1)}, \ldots, l'_{i\pi_i(M)}) \) is positive. Thus, for any goods allocation \( a \), there is a favored Leontief allocation, denoted \( L^\pi(a) \), defined by providing each agent with the best Leontief share that can be assembled from his share of the goods—to be explicit, this is obtained by starting with \( a_{i*} \) as the available goods vector, and then, for \( k \) from 1 to \( M \), setting \( L^\pi(a)_{i\pi_i(k)} \)
to be the largest \( \alpha \) such that \((\text{available goods vector}) - \alpha \lambda^k) \in \mathbb{R}^M_+\), then subtracting \( \alpha \lambda^k \) from the available goods vector and iterating.

We say that agent \( i \) majorization-prefers allocation \( a \) to \( b \), written \( a >^M_i b \), if

\[
\text{for all } K = 1, \ldots, M : \quad \sum_{k=1}^{K} L^\pi(a)_{i\pi(k)} \geq \sum_{k=1}^{K} L^\pi(b)_{i\pi(k)},
\]

with at least one of the inequalities being strict.

**The two orders.** Observe that “lexicographic-prefers” is a complete preference relation without indifference contours (since it is antisymmetric for distinct allocation shares), and that “majorization-prefers” is an incomplete preference relation; moreover the lexicographic order is a refinement of the majorization order, i.e., majorization-prefers implies lexicographic-prefers. The phrase “agent \( i \) weakly X-prefers” will be used to include the possibility that agent \( i \)'s share is identical in the two allocations.

1.1. **Our results.** The SG mechanism is deterministic, treats all agents symmetrically, and has the following properties.

**Properties w.r.t. majorization preference:**
- If all \( r_i \)'s are equal, the allocation produced by the SG mechanism in response to truthful bids is envy-free in the following sense: each agent weakly majorization-prefers his allocation to that of any other agent. This holds also in the Leontief case.

**Properties w.r.t. lexicographic preference:**
- The allocation produced by the SG mechanism in response to truthful bids is Pareto efficient. This holds also in the Leontief case.
- Incentive compatibility for a single agent: In the non-Leontief case, the SG mechanism is strategy-proof if \( \min_j q_j \geq \max_i r_i \).
  - We give counterexamples (a) in the absence of this inequality, (b) for the Leontief case.
- Generalizing the previous item, we have: Incentive compatibility for a coalition: The SG mechanism is group strategy-proof against coalitions of \( \ell \) agents if \( \min_j q_j \geq \max_{|S|=\ell} \sum_{i \in S} r_i \).
- The running time to implement the SG mechanism is \( \tilde{O}(mn) \) in the non-Leontief case, and \( \tilde{O}(n(m^2 + M)) \) in the Leontief case.
- Any Pareto efficient allocation can be produced using a suitable “variable speeds” extension of the SG mechanism. This holds also in the Leontief case. (However, the variable speeds extension does not possess the rest of the properties listed above.)

The incentive compatibility properties are the main results of this paper.

1.2. **Literature.** There has been considerable work on the strategy-proof allocation of divisible goods in Arrow-Debreu economies, starting with the seminal work of Hurwicz [11], e.g., see [8, 13, 22, 23, 24, 26]. Most of these results are negative, among the recent ones being Zhou’s result showing that in a 2-agent, \( n \)-good pure exchange economy, there can be no allocation mechanism that is efficient, non-dictatorial (i.e., both agents must receive non-zero allocations) and strategy-proof [26].

The paper that is most closely related to our work is that of Bogomolnaia and Moulin [6]. In their setting there are \( n \) agents and \( n \) indivisible goods, each agent having a total preference ordering over the goods; the desired outcome is a matching of goods with agents. A straightforward mechanism for allocating one good to each agent is random priority (RP): pick a uniformly random permutation of the agents and ask each agent in turn to select a good among those left. It is easy
to see that this mechanism is \textit{ex post efficient}, i.e., the allocation it produces can be represented as a probability distribution over Pareto efficient deterministic allocations, and it is strategy-proof. However, it is not \textit{ex ante efficient}. A random allocation is said to \textit{ex ante efficient} if for any profile of von Neumann-Morgenstern utilities that are consistent with the preferences of agents, the expected utility vector is Pareto efficient. It is easy to see that \textit{ex ante efficiency} implies \textit{ex post efficiency}.

Solving a conjecture of Gale [9], Zhou [25] showed that no strategy-proof mechanism that elicits von Neumann-Morgenstern utilities and achieves Pareto efficiency can find a “fair” solution even in the weak sense of equal treatment of equals. He further showed that the solution found by RP may not be efficient if agents are endowed with utilities that are consistent with their preferences. Hence, \textit{ex ante efficiency} had to be sacrificed, if strategy-proofness and fairness were desired.

In the face of these choices, the work of Bogomolnaia and Moulin gave the notion of \textit{ordinal efficiency} that is intermediate between \textit{ex post} and \textit{ex ante} efficiency; an allocation \(a\) is ordinally efficient if there is no other allocation \(b\) such that every agent majorization-prefers \(b\) to \(a\). They went on to show that the mechanism called \textit{probabilistic serial} (PS), introduced in Crès and Moulin [7], yields an ordinally efficient allocation. Further they show that PS is envy-free and weakly strategy-proof, defined appropriately for the partial order “majorization-prefers”. Finally, Bogomolnaia and Moulin define an extension of PS by introducing different “eating rates” and show that this set of mechanisms characterizes the set of all ordinally efficient allocations.

Katta and Sethuraman [14] generalize the setting of Bogomolnaia and Moulin to the “full domain”, i.e., agents may be indifferent between pairs of goods. Thus, each agent partitions the goods by equality and defines a total order on the equivalence classes of her partition (the agent is equally happy with any good received from an equivalence class). For this setting, they give a randomized mechanism that is a generalization (different from ours) of PS and achieves the same game-theoretic properties as PS.

A mechanism that probabilistically allocates indivisible goods can also be viewed as one that fractionally allocates divisible goods. Under the latter interpretation, the SG mechanism is equivalent to PS for the case that \(m = n\) and the quantity of each good and the requirement of each agent is one unit. An important difference is that Bogomolnaia and Moulin analyze PS under an incomplete preference relation (majorization) in which “most” allocation shares are incomparable; whereas we analyze SG under a complete preference relation (lexicographic) that is a refinement of majorization. The statement that a mechanism’s allocation is Pareto efficient w.r.t. lexicographic preferences is considerably stronger than the same statement w.r.t. majorization preferences, because each agent’s share is dominated by more alternative shares in the lexicographic order, than it is in the majorization order; so, fewer allocations are Pareto efficient in the lexicographic than in the majorization order. Our results should be viewed therefore as demonstrating that the PS mechanism and its natural generalization, SG, have far stronger game-theoretic properties than even envisioned in [6].

For somewhat related questions primarily regarding regarding exchange economies, see Barberà and Jackson [4], Nicolo [19], Ghodsi et al. [10], and Li and Xue [16]. Finally, we remark only that the problem of allocating a \textit{single} divisible good among multiple agents with known privileges is considerably different; the principal issue studied in that problem is how to make the division in a manner that is fair w.r.t. the given privileges. This is known as the bankruptcy problem and has a long history, e.g., see [21, 2]. Despite an interesting resemblance between the PS mechanism and some of the mechanisms used in the solutions of that problem [12], the issues at stake in the bankruptcy literature are distinct from those in our paper and its predecessors.
2. The Synchronized Greedy Mechanism

The mechanism is simple. Each agent \( i \) submits a preference list \( \sigma_i \). The submitted list may or may not, of course, agree with his true preference list \( \pi_i \).

(A simple case to consider is that of \( M = m = n \) and all \( q_j = r_i = 1 \). Because of the restriction that each preference list must include all \( m \) singleton bundles, each agent’s preference list in this case is a permutation of the \( m \) goods. Despite being quite special, this case, or the slightly more general case in which \( M = m \leq n \) and all \( r_i \) are equal, is already interesting to analyze and is well motivated by the examples, mentioned earlier, involving sharing of tasks or of scarce public resources.)

The mechanism simulates the following physical process. Consider each good \( j \) as a “liquid”, and each agent as a receptacle of capacity \( r_i \). The mechanism starts out at time 0 by (for all \( i \) in parallel) pouring bundle \( \lambda^{\sigma_i (1)} \) into receptacle \( i \) at rate \( r_i \) units of liquid per unit time. Each good \( j \) is therefore being drained at rate \( \sum_i r_i \lambda^{\sigma_i (1)}_j \). (Note that since \( \sum_j \lambda_j = 1 \), the total liquid being added to receptacle \( i \) per unit time is \( r_i \), as desired.)

This continues until one of the goods, say \( j \), is exhausted. For all agents who were currently being allocated bundles with \( j \) in their support, their favorite Leontief bundle has now been exhausted. (We say that a Leontief bundle has been exhausted at a given time if any of the goods in its support has been exhausted, and otherwise that the bundle is available.) All such agents, \( i \), are immediately allocated the next available bundle on their preference list, and the pouring of bundles continues. The algorithm continues in this way, allocating to an agent from the next available bundle whenever the current bundle has been exhausted. Since the singleton bundles are included in all preference lists, all agents continuously receive goods at rate \( r_i \) until time 1, at which time they simultaneously complete their full allocation.

Observe that the Leontief allocation \( l \) constructed by SG satisfies \( l = L^\pi (A(l)) \) because the bundles are provided to each agent greedily based on the availability of goods.

This continuous process can easily be converted into a discrete algorithm with the runtime cited earlier: maintain a priority queue of goods, keyed by termination times. Each time a good \( j \) is exhausted, each agent is assigned its next unexhausted bundle, and an updated termination time for each good is computed using the coefficients of the active bundles.

Observe that if an agent prefers bundle \( \lambda \) to bundle \( \lambda' \), and \( \text{support}(\lambda) \subseteq \text{support}(\lambda') \), then \( \lambda' \) may be removed from the agent’s preference list. It cannot be allocated to the agent by SG nor can it be part of any Pareto efficient allocation to the agent.

3. Properties of the Synchronized Greedy Mechanism

3.1. Pareto Efficiency. Let \( l^\sigma \) be the allocation created by the SG mechanism in response to bids \( \sigma \) declared by the agents. As before \( \pi \) denotes the truthful bids.

**Theorem 1.** The allocation produced by the SG mechanism in response to truthful bids is Pareto efficient w.r.t. lexicographic preference. That is to say, for all \( l \neq l^\pi \), \( \exists i \ l_i < l^\pi_i \).

**Proof.** For agent \( i \) and for \( K \geq 1 \) let \( t_{iK} = \frac{1}{r_i} \sum_{k=1}^{K} l_i^{\pi_i (k)} \). If agent \( i \) receives a positive quantity of his \( K \)’th-most-favored bundle, then \( t_{iK} \) is the time when that bundle is exhausted in SG. If the agent receives nothing from the bundle then the bundle is exhausted in SG no later than \( t_{iK} \).

Suppose for contradiction the existence of \( l \) s.t. \( \forall i \ l_i \geq l^\pi_i \), and for some \( i \), \( l_i > l^\pi_i \). Let \( t \) be minimum s.t. \( \exists i, K \ s.t. \ t = t_{iK} < \frac{1}{r_i} \sum_{k=1}^{K} l_i^{\pi_i (k)} \). Note, if \( t_{i'K'} < t \) then \( t_{i'K'} = \frac{1}{r_i} \sum_{k=1}^{K'} l_i^{\pi_i (k)} \).

For every one of the bundles \( b \in \{ \pi_i (1), \ldots, \pi_i (K) \} \) there is a good \( j(b) \) that appears positively in \( b \) and which is exhausted by time \( t \). Since \( t_{iK} < \frac{1}{r_i} \sum_{k=1}^{K} l_i^{\pi_i (k)} \) while \( t_{iK'} = \frac{1}{r_i} \sum_{k=1}^{K'} l_i^{\pi_i (k)} \) for all \( K' < K \), some agent \( i' \neq i \) receives strictly less of good \( j(\pi_i (K)) \) in \( l \) than in \( l^\pi \). Since \( j(\pi_i (K)) \)
is exhausted in SG by time $t$, this means that there is some $K''$ such that $rac{1}{r_i} \sum_{k=1}^{K''} l_{i'} \pi'_{i'}(k) < \frac{1}{r_i} \sum_{k=1}^{K''} l_{i'} \pi_{i'}(k) \leq t$. This contradicts the minimality of $t$. \hfill $\Box$

3.2. Strategy-Proofness. A mechanism is said to be strategy-proof if for every agent and for every list of bids by the remaining agents, the agent cannot obtain a strictly improved allocation by lying.

Theorem 2. In the non-Leontief case, the SG mechanism is strategy-proof if $\min q_j \geq \max r_i$.

Proof sketch: Consider two runs of SG, the first with the original bids and the second with agent $i$’s false bid and all other bids same. Clearly, in the latter, $i$ will demote some good in the preference order. Let $A$ be her most preferred good that she demotes. Assume that in the first run $i$ started getting good $A$ at time $t_A$. Since the amount of $A$ available is at least as large as the requirement of $i$ and since $i$’s allocation is not entirely composed of $A$, there is another agent, say $j$, who also got $A$ in the first run.

Let $G_j$ be the goods that $j$ prefers to $A$. At any time $t \geq t_A$, and until $A$ is exhausted in the second run, the total amount of the goods $G_j$ still unallocated is (weakly) more in the first run than in the second run– the reason is that $i$ starts consuming other goods instead of $A$ in the second run. Therefore, in the second run, $j$ must start receiving $A$ at least as early as in the first run. This holds for all agents who get $A$ in the first run. Furthermore, since agent $i$ starts receiving $A$ later in the second run compared to the first run, she will get less of $A$, thereby getting an inferior allocation in the second run.

Proof. Without loss of generality focus on agent 1. For the remainder of this proof $\pi_2, \ldots, \pi_n$ are arbitrary bids by the agents 2, \ldots, $n$, but $\pi_1$ is agent 1’s truthful bid. We need to show that for any bid $\sigma_1$ (and write $\sigma = (\sigma_1, \pi_2, \ldots, \pi_n)$), $a_{1s}^\sigma \leq a_{1s}^\pi$. The theorem is trivial if $a_{1s}^\sigma = a_{1s}^\pi$.

The theorem is also trivial if agent 1, bidding truthfully, receives only his top choice. So we may suppose that agent 1 does not receive the entire allocation of any one good.

We may also suppose that if $a_{1j}^\sigma = 0$ and $a_{1j'}^\sigma > 0$, then $\sigma_1(j) > \sigma_1(j')$. In other words, all the requests in $\sigma_1$ that come up empty may as well be deferred to the end.

Let $\pi_1^{-1}(j)$ be the $s$ such that $\pi_1(s) = j$. Let $G(j) = \{j' : \pi_1^{-1}(j') \leq \pi_1^{-1}(j) \text{ and } a_{1s}^\sigma(j') > 0\}$. Say that agent 1 sacrifices good $j$ in $\sigma$ if:

(1) $a_{1j}^\sigma > 0$,
(2) $\sigma_1(j) > |G(j)|$, and
(3) $\pi_1^{-1}(j) < \pi_1^{-1}(j')$ if $j'$ also satisfies (1),(2).

That is to say, $j$ is the most-preferred good which agent 1 receives a positive quantity in $\pi$, but requests later in $\sigma$ than in $\pi$.

Agent 1 must sacrifice some good, call it $B$, since otherwise the allocation will not change. See Figure 1. We will show that agent 1 receives strictly less of $B$ in $\sigma$ than in $\pi$, and that this is not compensated by higher-ranked goods.

Lemma 3. If $D$ is a good and $T_D^\pi < T_B^\pi$, then $T_D^\sigma \leq T_B^\sigma$.

Proof. Supposing the contrary, let $D$ be a counterexample minimizing $T_B^\sigma$. Since $T_D^\pi < T_B^\pi$, $D \neq B$.

Now let $i$ be any agent (who may or may not be agent 1) for whom $a_{iD}^\pi > 0$. Due to the minimality of $D$, each of the goods $j$ which $i$ prefers in $\pi$ to $D$, has $T_D^\sigma \leq T_D^\pi$. Therefore $i$ requests $D$ at a time in $\sigma$ that is at least as soon as the time $i$ requests it in $\pi$.

Since this holds for all $i$ who received a positive allocation of $D$ in $\pi$, the lemma follows. \hfill $\Box$

Let $N_B$ be the set of agents $i \neq 1$ for whom $a_{iB}^\pi > 0$. Due to the lemma, for each agent in $\{1\} \cup N_B$, the request time for $B$ in $\sigma$ is weakly earlier than it is in $\pi$. Now let $C$ be the good
Proposition 4. If \( \pi_1^{-1}(j') \leq \pi_1^{-1}(C) \), then \( a_{1j'}^{\pi} = a_{1j}^{\pi} \).

Proof. Supposing the contrary, let \( \pi_1^{-1}(j') \) be minimal such that \( \pi_1^{-1}(j') \leq \pi_1^{-1}(C) \) and \( a_{1j'}^{\pi} \neq a_{1j}^{\pi} \).

There are two possibilities to consider.

(a) \( a_{1j'}^{\pi} < a_{1j}^{\pi} \). This is not possible because then \( a_{1*}^{\pi} < a_{1*}^{\pi} \).

(b) \( a_{1j'}^{\pi} > a_{1j}^{\pi} \). Note:

Lemma 5. Let \( j_1, j_2 \) be such that \( \pi_1^{-1}(j_1) \leq \pi_1^{-1}(B) \), \( \pi_1^{-1}(j_2) \leq \pi_1^{-1}(B) \), \( a_{1j_1}^{\pi} > 0 \), and \( \pi_1^{-1}(j_1) < \pi_1^{-1}(j_2) \). Then \( \sigma_1^{-1}(j_1) < \sigma_1^{-1}(j_2) \).

Proof. Consider the least \( j_1 \) that is part of a pair \( j_1, j_2 \) violating the lemma. Then \( j_1 \) satisfies conditions (1),(2) above, contradicting that \( B \) is the good sacrificed by agent 1.

It follows that \( T_{j'}^{\sigma} \geq \sum_{j'' : \pi_1^{-1}(j'') \leq \pi_1^{-1}(j')} a_{1j''}^{\sigma} \). Due to the minimality of \( j' \), this means that if \( a_{1j'}^{\sigma} > a_{1j}^{\sigma} \), then \( T_{j'}^{\sigma} > T_{j'}^{\sigma} \), contradicting our earlier conclusion. This completes demonstration of the Proposition.

A consequence of the Proposition is that \( T_{C}^{\sigma} = T_{C}^{\sigma} \).

Since agent 1 sacrifices \( B \), his request time for \( B \) in \( \sigma \) is strictly greater than his request time for \( B \) in \( \pi \).

Recall that \( N_B \) is nonempty. At time \( T_B^{\sigma} \), the agents of \( N_B \) have received as least as much of \( B \) in \( \sigma \) as they have in \( \pi \), and the latter is positive. On the other hand, at the same time \( T_B^{\sigma} \), agent 1 has received strictly less of \( B \) in \( \sigma \) than he has in \( \pi \). In order for agent 1 to receive at least as much of \( B \) in \( \sigma \) as in \( \pi \), he would have to receive all of \( B \) that is allocated after time \( T_B^{\sigma} \); however, that is not possible, because the set of agents receiving \( B \) after \( T_B^{\sigma} \) includes \( N_B \). Thus \( a_{1*}^{\pi} < a_{1*}^{\pi} \).

3.3. Necessity of a Hypothesis on \( \{r_i\}, \{q_j\} \). We next provide an example in which strategy-proofness fails in the absence of the condition \( \max r_i \leq \min q_j \). For convenience now let \( r_1 \geq \ldots \geq r_m \) and \( q_1 \leq \ldots \leq q_m \).
Example 6. Let $n = 2$ and $m = 3$. Let $r_1 = r_2 = 3/2$; label the goods $A, B, C$, let $q_A = q_B = q_C = 1$, and let the preference lists be $\pi_1 = (A, B, C)$, $\pi_2 = (B, C, A)$. If agent 1 bids truthfully he receives the sorted allocation $(1, 0, 1/2)$. If instead he bids $(B, A, C)$ (while agent 2 bids truthfully), he receives the improved sorted allocation $(1, 1/2, 0)$. See Figure 2.

This example does not limit the theorem sharply, because it uses $r_1 = (3/2)q_1$ rather than $r_1$ arbitrarily close to $q_1$. Jeremy Hurwitz has pointed out that one may construct similar examples with whenever $r_1 \geq q_1/(1 - q_2/\sum q_j)$; this would appear to be a tight bound.

3.4. **Failure of strategy-proofness for the Leontief case.** Theorem 2 has no equivalent for general Leontief bundles. Consider the following four-agent system with $r_1 = r_2 = r_3 = r_4 = 1$ and three goods in supply $q_A = q_B = 1, q_C = 2$. Agent 1’s desired Leontief bundles are in the preference order $(A, B, C)$ (this agent is interested only in singleton bundles); agent 2 and 3’s desired Leontief bundles are in the order $(A, B, C, A, B)$; agent 4’s Leontief bundles are in the order $(B, C, A)$.

Under truthful bidding agent 1 receives the sorted goods allocation $(1/2, 0, 1/2)$. By bidding instead $(B, A, C)$, agent 1 receives the improved sorted goods allocation $(2/3, 1/3, 0)$. See Figure 3.

3.5. **Group Strategy-Proofness.** A mechanism is group strategy-proof against a family $F$ of subsets of agents if for every “coalition” $S \in F$ and for any list of bids by the agents outside of $S$, ...
the agents of \( S \) cannot obtain an improved allocation by lying, where by “improved allocation” we mean that no agent of \( S \) obtains a worse allocation and at least one obtains a strictly better allocation.

We now provide the following generalization of Theorem 2:

**Theorem 7.** In the non-Leontief case, the SG mechanism is group strategy-proof against the family of subsets \( S \) for which \( \min_j q_j \geq \sum_{i \in S} r_i \).

**Corollary 8.** In the non-Leontief case, the SG mechanism is group strategy-proof against coalitions of \( \ell \) agents if \( \min_j q_j \geq \max_{S, |S| = \ell} \sum_{i \in S} r_i \).

The proof of Theorem 7 follows a structure similar to that of Theorem 2 but the argument is complicated by the fact that different agents in \( S \) can sacrifice different goods, and some of the agents may actually be better off due to their untruthful bids (as they may benefit from the interactions among the several lies). The proof needs to effectively “chase through” an unbounded iteration of good transfers relative to \( a^\pi \), and show that some agent in the coalition is worse off than in \( \pi \). Fortunately, this can be done without explicitly pursuing the iteration.

**Proof.** Let \( S \) be a minimal counterexample. That is,

(a) \( \min_j q_j \geq \sum_{i \in S} r_i \);
(b) With \( \pi_i \) representing in this proof the truthful preferences for \( i \in S \) and arbitrary preferences for \( i \notin S \), there are bids \( \sigma_i \) for \( i \in S \) such that every \( i \in S \) “is a willing participant in the coalition \( S' \), namely (with \( \sigma_j = \pi_j \) for \( j \notin S \)) \( a^\sigma_{ij} \geq a^\pi_{ij} \);
(c) For some \( i \in S \), \( a^\sigma_{ij} > a^\pi_{ij} \);
(d) No strict subset of \( S \) satisfies (a),(b),(c).

Note by minimality that in \( \sigma \), every agent \( i \in S \) bids untruthfully (differently from \( \pi \)) and this has an effect, namely, if \( i \) reverts to bidding according to \( \pi \) then the allocation is different than in \( \sigma \).

If \( a^\sigma_{i\pi_i(1)} = r_i \) for all \( i \in S \), that is, with truthful bids these agents receive only their top choices, then none of them can be strictly rewarded by submitting a different bid.

Otherwise (i.e., if \( a^\sigma_{i\pi_i(1)} < r_i \) for some \( i \in S \)), then thanks to the hypothesis, under the truthful bids \( \pi \), every good has a positive allocation outside \( S \).

We may simplify the argument slightly by supposing that for each agent \( i \in S \), if \( a^\sigma_{ij} = 0 \) and \( a^\pi_{ij} > 0 \), then \( \sigma_i(j) > \sigma_i(j') \). In other words, all the requests that come up empty may as well be deferred to the end.

Let \( \pi_i^{-1}(j) \) be the \( s \) such that \( \pi_i(s) = j \). Let \( G(i,j) = \{j' : \pi_i^{-1}(j') \leq \pi_i^{-1}(j) \text{ and } a^\pi_i(j') > 0\} \).

Say that agent \( i \) sacrifices good \( j \) in \( \sigma \) if:

1. \( a^\pi_{ij} > 0 \),
2. \( \sigma_i(j) > |G(i,j)| \), and
3. \( \pi_i^{-1}(j) < \pi_i^{-1}(j') \) if \( j' \) also satisfies (1),(2).

Some good must be sacrificed by some agent, since otherwise the allocation will not change. (However, while every agent in \( S \) is untruthful, not every \( i \in S \) necessarily sacrifices a good; setting \( \sigma_i(j) > \pi_i(j) \) might have an effect even if \( a^\pi_i(j) = 0 \) because of increased availability of \( j \) due to bidding changes of other agents.)

Of all the sacrificed goods let \( B \) be one for which \( T^\sigma_B \) is minimal.

**Lemma 9.** If \( D \) is a good and \( T^\pi_D < T^\pi_B \), then \( T^\sigma_D \leq T^\sigma_B \).

**Proof.** Supposing the contrary, let \( D \) be a counterexample minimizing \( T^\sigma_B \). By the minimality of \( B, D \) cannot be a sacrificed good.
Now let $i$ be any agent (inside or outside of $S$) for whom $a_{iB}^\sigma > 0$. Due to the minimality of $D$, each of the goods $j$ which $i$ truthfully prefers to $D$, has $T_j^\sigma > T_j^\pi$. Therefore $i$ requests $D$ at a time in $\sigma$ that is at least as soon as the time $i$ requests it in $\pi$.

Since this holds for all $i$ who received a positive allocation of $D$ in $\pi$, the lemma follows. □

Let $O_B \subseteq S$ be the set of agents who sacrifice $B$, and let $N_B$ be the set of agents $i$ for whom $a_{iB}^\sigma > 0$ but who do not sacrifice $B$. Due to the lemma, for each agent in $O_B \cup N_B$, the request time for $B$ in $\sigma$ is weakly earlier than it is in $\pi$. Now consider an agent $i \in O_B$. Let $C$ be the good such that $\pi_i^{-1}(C)$ is maximal subject to $\pi_i^{-1}(C) < \pi_i^{-1}(B)$ and $a_{ij}^\sigma(C) > 0$. Due to the lemma, all goods $j'$ such that $\pi_i^{-1}(j') \leq \pi_i^{-1}(C)$ have $T_j^\sigma \leq T_j^\pi$. Next we show:

**Proposition 10.** If $\pi_i^{-1}(j') \leq \pi_i^{-1}(C)$, then $a_{ij'}^\sigma = a_{ij'}^\pi$.

**Proof.** Supposing the contrary, let $\pi_i^{-1}(j')$ be minimal such that $\pi_i^{-1}(j') \leq \pi_i^{-1}(C)$ and $a_{ij'}^\sigma \neq a_{ij'}^\pi$. There are two possibilities to consider.

(a) $a_{ij'}^\sigma < a_{ij'}^\pi$. This is not possible because $i$ is a willing participant in the coalition.

(b) $a_{ij'}^\sigma > a_{ij'}^\pi$. Note:

**Lemma 11.** Let $j_1, j_2$ be such that $\pi_i^{-1}(j_1) \leq \pi_i^{-1}(B), \pi_i^{-1}(j_2) \leq \pi_i^{-1}(B), a_{ij_1}^\sigma > 0$, and $\pi_i^{-1}(j_1) < \pi_i^{-1}(j_2)$. Then $\sigma_i^{-1}(j_1) < \sigma_i^{-1}(j_2)$.

**Proof.** Identical to the proof of Lemma 5 with agent $i$ in place of agent 1. □

It follows that $T_j^\sigma \geq \sum_{j'' : \pi_i^{-1}(j'') \leq \pi_i^{-1}(j')} a_{ij''}^\sigma$. Due to the minimality of $j'$, this means that if $a_{ij'}^\sigma > a_{ij'}^\pi$, then $T_j^\sigma < T_j^\pi$, contradicting our earlier conclusion. This completes demonstration of the Proposition. □

A consequence of the Proposition is that $T_i^\sigma = T_i^\pi$.

Since agent $i$ sacrifices $B$, his request time for $B$ in $\sigma$ is strictly greater than his request time for $B$ in $\pi$.

Since we are in the case that every good has a positive allocation outside $S$, $N_B$ is nonempty. At time $T_B$, the agents of $N_B$ have received as least as much of $B$ in $\sigma$ as they have in $\pi$, and the latter is positive. On the other hand, at the same time $T_B$, the agents of $O_B$ have received strictly less of $B$ in $\sigma$ than they have in $\pi$. In order for the agents of $O_B$ to receive collectively at least as much of $B$ in $\sigma$ as in $\pi$, they would have to receive all of $B$ that is allocated after time $T_B$; however, that is not possible, because the set of agents receiving $B$ after $T_B$ includes $N_B$. Therefore there is some $i \in O_B$ for whom $a_{iB}^\sigma < a_{iB}^\pi$. This contradicts the requirement that $i$ be a willing participant in the coalition $S$. □

**Example 12.** Example 6, in which strategy-proofness failed absent the hypothesis of Theorem 2, can be extended in a straightforward manner to one in which the group strategy-proof property fails to hold absent the hypothesis of Corollary 8. Again use $m = 3$, but instead of two agents, use $n = 2\ell$ agents, the first half having the same preference order $(A, B, C)$ as agent 1 in the earlier example, and the second half having the same preference order $(B, C, A)$ as agent 2 in the earlier example. If all agents bid truthfully, then the first $\ell$ agents each receive the sorted allocation $(1, 0, 1/2)$; however if they lie and bid $(B, A, C)$, while the remainder bid truthfully, then each lying agent receives the improved sorted allocation $(1, 1/2, 0)$.

## 4. Characterizing All Pareto Efficient Allocations

Bogomolnaia and Moulin [6] extended their mechanism by allowing players to receive goods at time-varying rates. Specifically, for each agent $i$ there is a speed function $\eta_i$ mapping the time
interval $[0, 1]$ into the nonnegative reals, such that for all $i$

$$
\int_0^1 \eta_i(t) \, dt = r_i.
$$

Subject to these speeds, goods flow to agents in order of the preference lists they bid, just as before. They showed that this extension characterizes all ordinally efficient allocations.

In this section, we obtain an analogous characterization of all Pareto efficient allocations by a similar extension of our mechanism. Specifically, we prove that for any Pareto efficient allocation of bundles, there exist speeds such that the extended SG mechanism produces that allocation. We prove this after first noting that the extended SG mechanism always results in Pareto efficient allocations.

In this section when $\eta_i \ (1 \leq i \leq n)$ are fixed, we let $a^\pi$ (with the $\eta$’s implicit) be the goods allocation produced by the extended SG mechanism with these speeds and truthful bids. We let $l^\pi = L^\pi(a^\pi)$ be the corresponding allocation of bundles.

4.1. Pareto Efficiency.

**Theorem 13.** Let $\eta_i, 1 \leq i \leq n$, be any speed functions. Then the allocation $l^\pi$ is Pareto efficient.

**Proof.** The argument is the same as for Theorem 1 with the proviso that the definition $t_{iK} = \frac{1}{\lambda_i} \sum_{k=1}^K l^\pi_{i\pi_i(k)}$ is replaced by $t_{iK} = \inf \{ y : \int_0^y \eta_i(t) \, dt \geq \sum_{k=1}^K l^\pi_{i\pi_i(k)} \}$. \hfill\Box

4.2. Characterizing All Pareto Efficient Allocations. If the last result mirrored the First Welfare Theorem, the next mirrors the Second Welfare Theorem:

**Theorem 14.** Let $\pi$ be the collection of agent preference lists over bundles, and let $l$ be a Pareto efficient allocation. There exist speed functions $\eta_i, 1 \leq i \leq n$, such that $l = l^\pi$.

**Proof.** As before the bundles are $(\lambda^k)_k \in \mathbb{N}$, where for each $k$, $\sum_{j=1}^m \lambda^k_j = 1$, and $\lambda^k_j \geq 0$ for all $j$.

Construction of the speeds $\eta_i$ is simple. Let a “partial bundle allocation” be a list $\hat{l}_{ik}$, each $\hat{l}_{ik} \geq 0$, such that for every $i$, $\sum_{k,j} \hat{l}_{ik} \lambda^k_j \leq r_i$, and for every $j$, $\sum_{i,k} \hat{l}_{ik} \lambda^k_j \leq q_j$.

Initialize $t = 0$ and initialize each agent $i$ with the empty partial allocation $\hat{l}_{ik} = 0$ for all $i, k$.

Initialize $c_j$ to be the quantity of good $j$ that is allocated in $l$. (Necessarily $c_j \leq q_j$ and $\sum c_j = \sum r_i$. If $\sum q_j > \sum r_i$ then for some $j$, $c_j < q_j$.)

Then repeat the following until $t = 1$.

Find an agent $i$ for whom there is an $\ell$ such that $\hat{l}_{i\pi_i(\ell)} < l_{i\pi_i(\ell)}$, and such that for all $\ell' < \ell$, the bundle $\pi_i(\ell')$ has been exhausted (that is to say, there is a good $j$ such that $\lambda^\pi_i(\ell') > 0$ and $c_j = 0$.) To see that there is such an $i$, suppose the contrary, and consider all the agents for whom $\sum_{k,j} \hat{l}_{ik} \lambda^k_j < r_i$. For each of them there is a favorite bundle which has not yet been exhausted. Evidently none of these agents is to be allocated in $l$ any additional quantity of this favorite bundle. However since these favorite bundles have not yet been exhausted, we can allocate to every player a slight additional positive amount of his favorite unexhausted bundle, without exhausting any additional goods. Any extension of this new partial allocation to a full bundle allocation, strictly Pareto dominates $l$, contrary to assumption.

Now set $\delta = (l_{i\pi_i(\ell)} - \hat{l}_{i\pi_i(\ell)})/\sum r_i$. For $t < t' < t + \delta$, make the settings $\eta_i(t') = \sum r_i$ and, for $i' \neq i$, $\eta_{i'}(t') = 0$. Then increment $\hat{l}_{i\pi_i(\ell)}$ by $\delta \sum r_i$, and decrement each $c_j$ by the corresponding amount, namely, decrement $c_j$ by $\lambda^\pi_i(\ell') \delta \sum r_i$. Finally, increment $t$ by $\delta$.

This process terminates in finitely many iterations because in each iteration some agent completes its allocation of some bundle. \hfill\Box

Examination of the above proof reveals:
Corollary 15. There is a polynomial time algorithm for checking whether a given allocation is Pareto efficient.

4.3. No Incentive Compatibility for the Variable Speeds Variant. We note that the synchrony imposed among agents by the SG mechanism is key to its incentive compatibility and envy-freeness properties (indeed, the properties hold even if the basic mechanism is extended with the same speed function for all agents). If different agents have different speed functions under the extended SG mechanism, Theorems 2 and 7, showing incentive compatibility, fail to hold. The argument breaks down as soon as it uses termination times, in Lemma 3. Below is a counter-example for strategy-proofness; a similar idea gives counter-examples for group strategy-proofness and envy-freeness.

Example 16. Assume \( m = n = 4 \) and that all \( r_i = q_j = 1 \). Let the speed function for agent 1 be 1 over the interval \([0, 1]\). The speeds of agents 2, 3, and 4 equal 1 over the interval \([1/2, 5/6]\), and 3 over the interval \([5/6, 1]\). The preference orders of agents 1 and 2 are \((1, 2, 3, 4)\), and the preference orders of agents 3 and 4 are \((2, 4, 3, 1)\). If all agents bid truthfully, agent 1 receives the sorted allocation \((1/2, 0, 1/2, 0)\). On the other hand, if agent 1 bids \((2, 1, 3, 4)\) while the rest bid truthfully, then agent 1 receives the better sorted allocation \((1/2, 1/3, 1/6, 0)\).

5. Envy-Freeness w.r.t. majorization preference

(This section is the only part of the paper where we use majorization preference.)

Given a bundle allocation \( l \), let \( \bar{l} \) denote the relative allocation, where \( \bar{l}_{ij} = l_{ij}/r_i \).

Theorem 17. Under truthful bidding, every agent \( i \) weakly majorization-prefers his relative allocation \( \bar{l}_{i\pi_i^*} \) to the relative allocation \( \bar{l}_{i'\pi_i^*} \) of any other agent \( i' \).

Proof. Fix any \( 1 \leq k \leq M \). We are to show that

\[
\frac{1}{r_i} \sum_{\ell=1}^{k} l_{i\pi_i(\ell)} \geq \frac{1}{r_{i'}} \sum_{\ell=1}^{k} l_{i'\pi_i(\ell)}.
\]

Let \( t \) be the time at which the last of the bundles \( \pi_i(1), \ldots, \pi_i(k) \) is exhausted. So \( tr_i = \sum_{\ell=1}^{k} l_{i\pi_i(\ell)} \). No other agent can receive any of these bundles after time \( t \), so \( tr_i' \geq \sum_{\ell=1}^{k} l_{i'\pi_i(\ell)} \). \( \square \)

6. Discussion

Our main open problem is the one mentioned in the Introduction, i.e., achieving approximate versions of the properties of the SG mechanism but when agents’ preferences are representable by utility functions.

In [5], Bogomolnaia and Heo show that efficiency (under the majorization relation), envy-freeness, and a property they call bounded invariance characterize the PS mechanism. This leads to the question of appropriately characterizing the SG mechanism. Towards this end we ask if efficiency, under the more stringent lexicographic relation, and envy-freeness suffice. Clearly, a first step would be to characterize the PS mechanism in this manner. A mechanism is said to have the bounded invariance property if for any agent \( i \) and any good \( j \), changing \( i \)'s preference order for goods she likes less than \( j \) does not change the amount (equivalently, probability) of good \( j \) each agent gets.

A natural open question concerns the existence of mechanisms to produce lexicographically most equitable allocations, having favorable algorithmic and game-theoretic properties (e.g., incentive compatibility).

As mentioned in the Introduction, Katta and Sethuraman [14] generalize the setting of Bogomolnaia and Moulin to the “full domain”, i.e., agents may be indifferent between pairs of goods,
and they give a randomized mechanism that is a generalization of PS and achieves the same game-theoretic properties as PS. We ask the analogous question for the generalization of our setting, i.e., each agent partitions the goods by equality and defines a total order on the equivalence classes of her partition (the agent is equally happy with any good received from an equivalence class). Preferences are again defined lexicographically over classes, i.e., by considering the total amount of goods received in each class. Is there a generalization of our mechanism to this setting?

Finally, one expects that allocation quality will increase with the diversity of agent preferences. A natural waystep to consider is the very diverse setting in which agent preferences are independent uniformly random permutations of 1, . . . , m. This suggests many interesting questions, e.g., what is the distribution of βk as in Eqn. A.2, for the allocation a given by the SG mechanism; and, what is the distribution of the maximized value of t as in LP (A.1), both for various values of k. Regarding the latter question, for n = m and all ri = qj = 1, there is a correspondence in the case k = 1 with the collision statistics of random pointers, and so it is known that t → (log log n)/(log n); for larger k there is a rough correspondence with the “power of two choices” literature [3, 18], suggesting likely asymptotics of (log k)/(log log n) for fixed k, although the correspondence between the problems is not close enough for us to state this with certainty.

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References

Appendix A. Equitability of Allocations

It is interesting to consider whether an allocation mechanism is equitable—minimizing, in some measurable sense, the disparity in the welfare of the players. In spite of being deterministic and treating all agents symmetrically, the SG mechanism is not particularly equitable, except as regards how much each agent receives of his most-preferred good. We provide an example showing that even the allocations of each agent’s two most preferred goods may be quite inequitable.

On the other hand we describe a time-efficient algorithm that, for any given $1 \leq k \leq m$, equitably allocates the top $k$ goods for each agent. We further define the notion of a *lexicographically most equitable* allocation and give a time-efficient algorithm to find one. In this section we consider only the non-Leontief case.

Recall that for an allocation $a$ let $\bar{a}$ denotes the relative allocation $\bar{a}_{ij} = a_{ij}/r_i$. For any $k$, $1 \leq k \leq m$, say that an allocation is equitable w.r.t. agents’ top $k$ choices if it belongs to

$$\arg\max_a \min_i (\bar{a}_{i\pi_1(1)} + \ldots + \bar{a}_{i\pi_{k}(k)}),$$

where the max is over all allocations $a$.

It is easy to see that the allocation produced by the SG mechanism is equitable for $k = 1$. However, as the following example illustrates, it is not equitable for $k = 2$, or larger values of $k$.

**Example 18.** Let $n = 2$, $m = 3$, $r_1 = r_2 = 1$, $q_1 = 1/2$, $q_2 = 5/6$, and $q_3 = 2/3$. Let the preference list of the first agent be $(1, 2, 3)$ and that of the second agent $(2, 3, 1)$. Then the SG mechanism gives sorted allocations of $(1/2, 1/6, 1/3)$ and $(2/3, 1/3, 0)$ respectively to the agents, so each receives $2/3$ of his total allocation from his top two choices. On the other hand, the sorted allocations $(1/2, 1/2, 0)$, $(1/3, 2/3, 0)$ are also feasible, and in this case each agent receives his entire allocation from his top two choices.

Next, we show that there is a polynomial-time algorithm which, given $k$, $(r_i)$, $(q_j)$ and the list of agent preferences, obtains an allocation that is equitable w.r.t. agents’ top $k$ choices. In fact
this allocation \( x = (x_{ij}) \) is the solution to the linear program given below, together with \( t \), the minimum over agents of the relative allocation from the agent’s top \( k \) goods.

\[
\text{(A.1)} \quad \begin{align*}
\text{Maximize} & \quad t \\
\text{Such that} & \quad \forall i : t \leq \frac{1}{r_i} \left( \sum_{\ell=1}^{k} x_{i\pi_i(\ell)} \right) \\
& \quad \forall i : \sum_{j=1}^{m} x_{ij} = r_i \\
& \quad \forall j : \sum_{i=1}^{n} x_{ij} \leq q_j \\
& \quad \forall i, \forall j : x_{ij} \geq 0
\end{align*}
\]

Finally, let us define the notion of the lexicographically most equitable allocation, which intuitively is an allocation that simultaneously optimizes for each \( k \), to the extent possible. For any allocation \( a \), and each \( k, \, 1 \leq k \leq m \), define

\[
\text{(A.2)} \quad \beta_k = \min_i (\overline{a}_{i\pi_i(1)} + \cdots + \overline{a}_{i\pi_i(k)}).
\]

Now, define a lexicographically most equitable allocation to be one that lexicographically maximizes \( (\beta_1, \ldots, \beta_m) \).

We now give a polynomial-time algorithm to find a lexicographically most equitable allocation—it involves solving \( m \) LPs derived from LP (A.1). The first LP simply computes \( \beta_1 \) by solving LP (A.1) for \( k = 1 \). Next, for each \( k, \, 2 \leq k \leq m \), add the following constraints to LP (A.1) and solve it to determine \( \beta_k \):

\[
\forall i, \forall 1 \leq h \leq k - 1 : \frac{1}{r_i} \left( \sum_{\ell=1}^{h} x_{i\pi_i(\ell)} \right) \geq \beta_h.
\]

Clearly, the last LP will yield a most equitable allocation.

**Example 19.** For the agents in Example 18, the lexicographically most equitable allocation is (given as a sorted allocation): \((1/2, 1/3, 1/6)\) for agent 1 and \((1/2, 1/2, 0)\) for agent 2. This is different from both the SG allocation and the allocation that is equitable w.r.t. agents’ top 2 choices.

Although equitability would seem to be a desirable property, it must be noted that an equitable allocation need not be even Pareto efficient:

**Example 20.** Let \( n = 3, \, m = 4, \, r_1 = r_2 = r_3 = 2, \, q_1 = q_2 = 1, \, q_3 = q_4 = 2 \). Let the preference lists be \( \pi_1 = (1, 2, 3, 4), \, \pi_2 = (3, 4, 1, 2), \, \pi_3 = (4, 3, 1, 2) \). For any \( 0 \leq x \leq 1 \), the following allocation is lexicographically most equitable, and even stronger, it simultaneously optimizes all \( \beta_k \) in Eqn. A.2:

\[
a_1 = (1, 1, 0, 0), \, a_2 = (0, 0, 2 - x, x), \, a_3 = (0, 0, x, 2 - x).
\]

Yet this allocation is Pareto efficient only in the single case \( x = 0 \).