[Cook-Levin '71] SAT is NP-complete.

Using this we showed:

\[
\begin{align*}
\text{SAT} & \quad \downarrow \\
\text{3SAT} & \quad \downarrow \\
\text{Independent Set} & \quad \downarrow \\
\text{Clique} & \quad \rightarrow \\
\text{Vertex Cover} & \quad \leftarrow \\
\end{align*}
\]

**3SAT:**

*Input:* Boolean $f$ in CNF with $n$ variables $x_1, \ldots, x_n$ and $2m$ clauses $C_1, \ldots, C_m$ where every $C_i$ has $|C_i| \leq 3$

*Output:* Satisfying assignment if one exists, NO otherwise.

Last class we saw: 3SAT is NP-complete.

**Independent-Set:**

*Input:* undirected $G=(V,E)$ & goal $g$.

*Output:* independent set $S$ where $|S| \geq g$, NO if no such $S$ exists.
Recall, an independent set $S$ is a subset $S \subseteq \mathcal{V}$ where no edges contained in $S$
i e, for all $x, y \in S$, $(x, y) \notin E$.

**Theorem:** Independent Set problem is NP-complete.

**Proof:**

a) $\text{IS} \in \text{NP}$: Given $G, g \& S$, in $O(n + m)$ time can check for every $(x, y) \in E$
that either $x \notin S$ \& $y \notin S$.
& in $O(n)$ time that $|S| \geq g$.

b) $3\text{SAT} \Rightarrow \text{IS}$

Consider input $f$ for $3\text{SAT}$.

We'll create a graph $G = (V, E)$
& we'll set the goal $g = m$.
The graph $G$ will have one vertex per literal per clause.

For example: for clause $C_i = (\overline{x_2} \lor \overline{x_4} \lor x_1)$, we create 3 new vertices, corresponding to $\overline{x_2}$, $\overline{x_4}$, & $x_1$.

We add a triangle between these 3 new vertices. Hence an independent set contains $\leq 1$ of them.

For clause of size 2, e.g., $C_i = (x_5 \lor \overline{x_6})$, then add 2 new vertices corresponding to $x_5$ & $\overline{x_6}$ & add edge between them.

And for clause of size 1: add 1 new vertex.
Finally, add edges between all $x_i$ & all $\overline{x_i}$ for all $i = 1, 2, \ldots, n$. (then any independent set contains some $x_i$'s or some $\overline{x_i}$'s but not both)

**Example:**

$$f = (x_1 \lor \overline{x_3} \lor x_4) \land (\overline{x_4}) \land (x_2 \lor x_4) \land (\overline{x_2} \lor x_3 \lor \overline{x_4})$$

Satisfying assignment:

- $x_1 = T$
- $x_2 = T$
- $x_3 = F$
- $x_4 = F$

**IS** $S = \{a, d, e, i\}$
f has a satisfying assignment \iff G has an independent set of size $g$.

**Proof:**

$(\Rightarrow)$ Given satisfying assignment for $f$ for each clause there's $\geq 1$ satisfied literal. Take one satisfied literal per clause & add those vertices to $S$.

Note $|S|=m$.

$S$ has exactly one vertex per clause so clause edges (triangles) are not included & we started from an assignment so $x_i=T$ or $x_i=F$ & thus $x_i \notin S$ vertices or $x_i \in S$ vertices can be in $S$ but not both so no edges $(x_i,x_i) \in S$.

$(\Leftarrow)$ Consider independent set $S$ of size $|S| \geq g = m$. 
$S$ has $\leq 1$ vertex per clause so $|S| \leq m$ & hence $|S| = m$.

For the vertex in clause $C_i$ satisfy the corresponding literal & this means $C_i$ is satisfied.

Since $G$ has edges $(x_i, \overline{x_i})$ thus $x_i = T$ or $x_i = F$ (or no setting)
so it's a valid assignment.