3SAT:

input: Boolean formula f in CNF with n variables & m clauses where each clause has ≤ 3 literals.

output: Satisfying assignment if one exists & NO otherwise.

Theorem: 3SAT is NP-complete.

We'll use the fact that SAT is NP-complete.

Need to show:

a) $3SAT \in NP$

b) $SAT \to 3SAT$.

For (a):

Given an assignment σ, in $O(1)$ time per clause we can check that ≥ 1 literal is satisfied. Thus $O(m)$ total time to check that f is satisfied.

For b: SAT $\to$ 3SAT.

Take input f for SAT.
We need to create input f' for 3SAT.
Then given a satisfying assignment for f'
we need to define a satisfying assignment
for f.
Finally, we need to show that:

$$f \text{ is satisfiable} \iff f' \text{ is satisfiable.}$$

Example: $f = \underbrace{(x_3)}_{C_1} \wedge \underbrace{(\overline{x_2} \vee x_3 \vee \overline{x_1} \vee \overline{x_4})}_{C_2} \wedge \underbrace{(x_2 \vee x_1)}_{C_3}$

Clauses C_1 & C_3 can stay the same but
C_2 is too big for 3SAT.

Create a new variable y.
Look at $C_2' = (\overline{x_2} \vee x_3 \vee y) \wedge (\overline{y} \vee \overline{x_1} \vee \overline{x_4})$

Claim: C_2 is satisfiable $\iff$ C_2' is satisfiable.

We'll prove a more general claim shortly.

What if C is of size 5?

Say $C = (\overline{x_2} \vee x_3 \vee \overline{x_1} \vee \overline{x_4} \vee x_5)$

Then create 2 new variables y_1 & y_2.

Let $C' = (\overline{x_2} \vee x_3 \vee y_1) \wedge (\overline{y_1} \vee \overline{x_1} \vee y_2) \wedge (\overline{y_2} \vee x_4 \vee x_5)$.

In general, for literals $a_1, \ldots, a_k$

for $C = (a_1 \vee a_2 \vee \cdots \vee a_k)$

add $k-3$ new variables $y_1, \ldots, y_{k-3}$

& replace C by:

$$C' = (a_1 \vee a_2 \vee y_1) \wedge (\overline{y_1} \vee a_3 \vee y_2) \wedge (\overline{y_2} \vee a_4 \vee y_3)$$
$$\wedge \cdots \wedge (\overline{y_{k-4}} \vee a_{k-2} \vee y_{k-3}) \wedge (\overline{y_{k-3}} \vee a_{k-1} \vee a_k)$$

Claim: C is satisfiable $\Leftrightarrow$ C' is satisfiable.

We'll prove something stronger:

~~C is satis~~

For any assignment to $a_1, \ldots, a_k$,

C is satisfied $\Longleftrightarrow$ there is an assignment to $y_1, \ldots, y_{k-2}$ so that C' is satisfied.

Thus, given a satisfying assignment to C' we ignore the assignment to the new variables & this same assignment for the original $x_1, \dots, x_n$ satisfies C.

<u>Proof:</u>

($\Rightarrow$) Take assignment to $a_1, \dots, a_i$ satisfying C.
$C = (a_1 \vee \dots \vee a_k)$ so at least 1 a_i is satisfied.
Let a_i be $\min_i$ where a_i is satisfied.

So $a_i = T \Rightarrow \underbrace{\text{clause } i-1}_{(\overline{y_{i-2}} \vee a_i \vee y_{i-1})}$ in C' is satisfied

Set $y_1 = y_2 = \dots = y_{i-2} = T$

$\Rightarrow$ 1st $i-2$ clauses in C' are satisfied.

Set $y_{i-1} = \dots = y_{k-2} = F$

$\Rightarrow$ clauses $i, \dots, k-2$ in C' are satisfied.

($\Leftarrow$) Take assignment to $a_1, \ldots, a_k, y_1, \ldots, y_{k-2}$ satisfying C'.

Suppose $a_1 = a_2 = \cdots = a_k = F$.

Then since C' is satisfied,

for clause 1 we must have that $y_1 = T$

for clause 2 " $y_2 = T$

⋮

for clause $k-3$ " $y_{k-3} = T$

& then the last clause $(\overline{y_{k-3}} \vee a_{k-1} \vee a_k)$ is not satisfied $\Rightarrow\Leftarrow$

So ≥ 1 of $a_1, \ldots, a_k$ is set to T.

& thus C is satisfied.

SAT → 3SAT:

Given f for SAT.

Create a new formula f' as follows:

For each clause C in f

let $k = |C| = \#$ of literals in C.

if $k \leq 3$ then add C to f'.

if $k > 3$ then:

create $k-3$ new variables
& replace C by C' as
described before.

Use f' as input for 3SAT.

We saw that f is satisfiable $\Leftrightarrow$ f' is satisfiable.

And given a satisfying assignment for f'
we ignore the new variables & we have
a satisfying assignment for f.