3SAT:
input: Boolean formula $f$ in CNF with $n$ variables and $m$ clauses where each clause has $\leq 3$ literals.
output: satisfying assignment if one exists & NO otherwise.

Theorem: 3SAT is NP-complete.
We'll use the fact that SAT is NP-complete.

Need to show:
  a) 3SAT $\in$ NP
  b) SAT $\rightarrow$ 3SAT.

For (a):
Given an assignment $\sigma$, in $O(1)$ time per clause we can check that $\geq 1$ literal is satisfied. Thus $O(m)$ total time to check that $f$ is satisfied.
For $b$: SAT $\rightarrow$ 3SAT.

Take input $f$ for SAT.

We need to create input $f'$ for 3SAT.

Then given a satisfying assignment for $f'$, we need to define a satisfying assignment for $f$.

Finally, we need to show that:

$f$ is satisfiable $\iff f'$ is satisfiable.

**Example:**

$$f = (x_3) \land (\overline{x_2} \lor x_3 \lor \overline{x_1} \lor \overline{x_4}) \land (x_2 \lor x_1)$$

Clauses $C_1$, $C_2$, $C_3$

Clauses $C_1$ & $C_3$ can stay the same but $C_2$ is too big for 3SAT.

Create a new variable $y$.

Look at $C_2' = (x_2 \lor x_3 \lor y) \land (\overline{y} \lor x_1 \lor \overline{x_4})$

Claim: $C_2$ is satisfiable $\iff C_2'$ is satisfiable.

We'll prove a more general claim shortly.
What if $C$ is of size 5?

Say $C = (\overline{x}_2 \lor x_3 \lor \overline{x}_1 \lor \overline{x}_4 \lor x_5)$

Then create 2 new variables $y_1$ & $y_2$.

Let $C' = (\overline{x}_2 \lor x_3 \lor y_1) \land (\overline{y}_1 \lor \overline{x}_1 \lor y_2) \land (\overline{y}_2 \lor x_4 \lor x_5)$.

In general, for literals $a_1, \ldots, a_k$

for $C = (a_1 v a_2 v \ldots v a_k)$

add $k-3$ new variables $y_1, \ldots, y_{k-3}$

& replace $C$ by:

$C' = (a_1 v a_2 v y_1) \land (\overline{y}_1 v a_3 v y_2) \land (\overline{y}_2 v a_4 v y_3) \land \ldots \land (\overline{y}_{k-4} v a_{k-2} v y_{k-3}) \land (\overline{y}_{k-3} v a_{k-1} v a_k)$

Claim: $C$ is satisfiable $\iff$ $C'$ is satisfiable.

We'll prove something stronger:

$C$ is satisfiable $\iff$ there is an assignment to $y_1, \ldots, y_{k-2}$ so that $C'$ is satisfiable.
Thus, given a satisfying assignment to $C'$ we ignore the assignment to the new variables & this same assignment for the original $x_1, \ldots, x_n$ satisfies $C$.

Proof:

(⇒) Take assignment to $a_1, \ldots, a_i$ satisfying $C$. $C = (a_1, v \ldots, v_{a_k})$ so at least 1 $a_i$ is satisfied. Let $a_i$ be min where $a_i$ is satisfied.

So $a_i = T \implies \text{clause } i-1 \text{ in } C'$ is satisfied

$(\neg y_{a_i} v a_i v y_{i-1})$

Set $y_1 = y_2 = \ldots = y_{i-2} = T$

⇒ $i$-th $i-2$ clauses in $C'$ are satisfied.

Set $y_{i-1} = \ldots = y_{k-2} = F$

⇒ clauses $i$, $\ldots$, $k-2$ in $C'$ are satisfied.
Take assignment to \( a_1, \ldots, a_k, y_1, \ldots, y_{k-2} \) satisfying \( C' \).

Suppose \( a_1 = a_2 = \ldots = a_{k-2} = F \).

Then since \( C' \) is satisfied,

for clause 1 we must have that \( y_1 = T \)

for clause 2 \( y_2 = T \)

\[ \vdots \]

for clause \( k-3 \) \( y_{k-3} = T \)

& then the last clause \((\overline{y_{k-3}} \lor a_{k-1} \lor a_k)\)

is not satisfied.

So \( \geq 1 \) of \( a_1, \ldots, a_k \) is set to \( T \).

& thus \( C \) is satisfied.
**SAT \rightarrow 3SAT:**

Given $f$ for SAT.

Create a new formula $f'$ as follows:

For each clause $C$ in $f$

let $k = |C| = \# \text{ of literals in } C$.

if $k \leq 3$ then add $C$ to $f'$

if $k > 3$ then:

create $k-3$ new variables

& replace $C$ by $C'$ as described before.

Use $f'$ as input for 3SAT.

We saw that $f$ is satisfiable $\Rightarrow f'$ is satisfiable.

And given a satisfying assignment for $f'$

we ignore the new variables & we have

a satisfying assignment for $f$. 